Optimal Pricing in Social Networks with Incomplete Information

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Abstract

In revenue maximization of selling a digital product in a social network, the utility of an agent is often considered to have two parts: a private valuation, and linearly additive influences from other agents. We study the incomplete information case where agents know a common distribution about others' private valuations, and make decisions simultaneously. The "rational behavior" of agents in this case is captured by the well-known Bayesian Nash equilibrium.

Two challenging questions arise: how to *compute* an equilibrium and how to *optimize* a pricing strategy accordingly to maximize the revenue assuming agents follow the equilibrium? In this paper, we mainly focus on the natural model where the private valuation of each agent is sampled from a uniform distribution, which turns out to be already challenging.

Our main result is a polynomial-time algorithm that can *exactly* compute the equilibrium and the optimal price, when pairwise influences are non-negative. If negative influences are allowed, computing any equilibrium even approximately is PPAD-hard. Our algorithm can also be used to design an FPTAS for optimizing discriminative price profile.

1 Introduction

Social influence in large social networks provides huge monetization potential, which is under intensive investigation by companies as well as research communities. Many digital products exhibit explicit social values. For example, Zune players can share music with each other, so the utility one can expect from a Zune player partially depends on the number of her friends having the same product. In a more direct case of instant messaging, the utility for one user is critically determined by the number of her friends who use the same instant messenger. Therefore, how to design, market, and price products with external social values depends intimately on the understanding and utilization of social influence in social networks.

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†A preliminary version of this work has appeared as a chapter of the B.Sci thesis of this author [Zhu10].

In this paper, we study the problem of selling a digital product to agents in a social network. To incorporate social influence, we assume each agent's utility of having the product is the summation of two parts: the private intrinsic valuation and the overall influence from her friends who also have the product. In this paper, we study the linear influence case, i.e., the overall influence is simply the summation of influence values from her friends who have the product.

Given such assumption, the purchasing decision of one agent is not solely made based on her own valuation, but also on information about her friends' purchasing decisions. However, a typical agent does not have complete information about others' private valuations, and thus might make the decision based on her belief of other agents' valuations.

In this paper, we study the case when this belief forms a public distribution, and rely on the solution concept of Bayesian Nash equilibrium [Har67]. Specifically, each agent knows her own private valuation (also referred to as her *type*); in addition, there is a distribution of this private valuation, publicly known by everyone in the network as well as the seller. In this paper, we study the case that the joint distribution is a product distribution, and the valuations for all agents are sampled independently from possibly different uniform distributions.

Computing the Equilibria. Usually, there exist multiple equilibria in this game. We first study the case when all influences are *non-negative*. We show that there exist two special ones: the *pessimistic equilibrium* and the *optimistic equilibrium*, and all other equilibria are between these two. We then design a polynomial time algorithm to compute the pessimistic (resp. optimistic) equilibrium *exactly*.

The overall idea is to utilize the fact that the pessimistic (resp. optimistic) equilibrium is "monotonically increasing" when the price increases. However, the iterative method requires exponential number of steps to converge, just like many potential games which may well be PLS-hard. Our algorithm is based on the line sweep paradigms, by increasing the price p and computing the equilibrium on the way. There are several challenges we have to address to implement the line sweep algorithm. See Section 3.2 for more discussions on the difficulties.

On the negative side, when there exist negative influences among agents, the monotone property of the equilibria does not hold. In fact, we show that computing an approximate equilibrium is PPAD-hard for a given price, by a reduction from the two player Nash equilibrium problem.

Optimal Pricing Strategy. When the seller considers offering a uniform price, our proposed line sweep algorithm calculates the equilibrium as a function of the price. This closed form allows us to find the price for the optimal revenue.

We also discuss the extensions to discriminative pricing setting: agents are partitioned into k groups and the seller can offer different prices to different groups. Depending on whether the algorithm can choose the partition or not, we discuss the hardness and approximation algorithms of these extensions.

1.1 Related Work

Influence maximization. Cabral et al. [CSW99] studied the property of the optimal prices over time with network externality and strategic agents. They show that the seller might set a low introductory price to attract a critical mass of agents. Another notable body of work in computer science is the *optimal seeding* problem (e.g. Kempe et al. [KKT03] and Chen et al. [CWY09]), in which a set of k seeds are selected to maximize the total influence according to some stochastic propagation model.

Pricing with equilibrium models. When there is social influence, a large stream of literature is focusing on simultaneous games. This is also known as the "two-stage" game where the seller sets the price in the first stage, and agents play a one-shot game in their purchasing decisions. Agents' rational behavior in this case is captured by the Nash equilibrium (or Bayesian Nash equilibrium if the information is incomplete).

The concept and existence of pessimistic and optimistic equilibria is not new. For instance, in analogous problems with externalities, Milgrom and Roberts [MR90] and Vives [Viv90] have witnessed the existence of such equilibria in the *complete information* setting. Notice that our pricing problem, when restricted to complete information, can be trivially solved by an iterative method.

In incomplete information setting, Vives and Van Zandt [VZV07] prove a similar existential result using iterative methods. However, they do not provide any convergence guarantee. In our setting, we have shown in Section 3.1 that such type of iterative methods may take exponential time to converge. Our proposed algorithm instead *exactly* computes the equilibrium, through a much move involved (but constructive) method. In parallel to this work, Sundararajan [Sun08] also discover the monotonicity of the equilibria, but for symmetry and limited knowledge of the structure (only the degree distribution is known).

It is worth noting that those works above do consider the case when influence is not linear (but for instance supermodular). Though our paper focuses on linear influences, our monotonicity results for equilibria do easily extend to non-linear ones. See Section 2.

When the influence is linear, Candogan, Bimpikis and Ozdaglar [CBO10] study the problem with (uniform) pricing model for a divisible good on sale. It differs from our paper in the model: they are in complete information and divisible good setting; more over, they have relied on a diagonal dominant assumption, which simplifies the problem and ensures the uniqueness of the equilibrium.

Another paper for linear influence is by Bloch and Querou [BQ09], which also studies the uniform pricing model. When the influence is small, they approximate the influence matrix by taking the first 3 layers of influence, and then an equilibrium can be easily computed. They also provide experiments to show that the approximation is numerically good for random inputs.

Pricing with cascading models. In contrast to the simultaneous-move game considered by us (and many others), another stream of work focuses on the cascading models with social influence.

Hartline, Mirrokni and Sundararajan [HMS08] study the *explore and exploit* framework. In their model the seller offers the product to the agents in a sequential manner, and assumes all agents are *myopic*, i.e., each agent is making the decision based on the known results of the previous agents in the sequence. As they have pointed out, if the pricing strategy of the seller and the private value distributions of the subsequent agents are publicly known, the agents can make more "informed" decisions than the myopic ones. In contrast to them, we consider "perfect rational" agents in the simultaneous-move game, where agents make decisions *in anticipation* of what others may do given their beliefs to other agents' valuations.

Arthur et al. [AMSX09] also use the explore and exploit framework, and study a similar problem; potential buyers do not arrive sequentially as in [HMS08], but can choose to buy the product with some probability only if being recommended by friends.

Recently, Akhlaghpour et al. [AGH+10] consider the multi-stage model that the seller sets different prices for each stage. In contrast to [HMS08], within each stage, agents are "perfectly rational", which is characterized by the pessimistic equilibrium in our setting with *complete information*. As mentioned in [AGH+10], they did not consider the case where a rational agent may defer her decision to later stages in order to improve the utility.

Other works. If the value of the product does not exhibit social influence, the seller can maximize the revenue following the optimal auction process by the seminal work of Myerson [Mye81]. Truthful auction mechanisms have also been studied for digital goods, where one can achieve constant ratio of the profit with optimal fixed price [GHK+06, HM05]. On computing equilibria for problems that guarantees to find an equilibrium through iterative methods, most of them, for instance the famous congestion game, is proved to be PLS-hard [FPT04].

2 Model and Solution Concept

We consider the sale of one digital product by a seller with zero cost, to the set of agents $V = [n] = \{1, 2, \ldots, n\}$ in a social network. The network is modeled as a simple *directed* graphs $G = (V, E)$ with no self-loops.

- **Valuation**: Agent i has a private value $v_i \geq 0$ for the product. We assume v_i is sampled from a uniform distribution with interval $[a_i, b_i]$ for $0 \leq a_i < b_i$, which we denote as $U(a_i, b_i)$. The values a_i and b_i are common knowledge.
- **Price**: We consider the seller offering the product at a uniform price p. We postpone discriminative pricing models in Appendix C.2.
- **Revenue:** Let $\mathbf{d} = \{d_1, \ldots, d_n\} \in \{0,1\}^n$ be the decision vector the agents make, i.e., $d_i = 1$ if agent i buys the product and 0 otherwise. The revenue of the seller is defined as $\sum_i p \cdot d_i$. When the decisions are random variables, the revenue is defined as the expected payments received from the users.
- **Influence**: Let matrix $\mathrm{T} = (\mathrm{T}_{j,i})$ with $\mathrm{T}_{j,i} \in \mathbb{R}$ and $i, j \in V$ represent the influences among agents, with $\mathrm{T}_{j,i} = 0$ for all $(j,i) \notin E$. In particular, $\mathrm{T}_{j,i}$ is the utility that agent i receives from agent j, if both of them buy the product. Except for the hardness result, we consider $\mathrm{T}_{j,i}$ to be non-negative.
- **Utility**: Let $\mathbf{d}_{-i}$ be the decision vector of the agents other than agent i. For convenience, we denote $\langle d_i', \mathbf{d}_{-i} \rangle$ the vector by replacing the i-th entry of $\mathbf{d}$ by d_i'. In particular, given the influence matrix T, the utility is defined as:

$$u_i(\langle d_i, \mathbf{d}_{-i} \rangle, v_i, p) = \begin{cases} v_i - p + \sum_{j \in [n]} d_j \cdot \mathrm{T}_{j,i}, & \text{if } d_i = 1 \\ 0, & \text{if } d_i = 0 \end{cases} \tag{1}$$

Remark 2.1. *In the definition, we require $a_i < b_i$. This condition can be relaxed to $a_i \leq b_i$, i.e., we are able to handle the fixed value case as well. For instance, this only requires a separate case analysis in our proposed line sweep algorithm in Section 3. However, for ease of presentation, we assume $a_i < b_i$ in the remaining of the paper, unless otherwise noted.*

Our question is: "which price shall the seller offer to maximize the total revenue?" In order to answer this question, it is necessary to study the agents' *rational behavior* using the concept Bayesian Nash equilibrium (BNE). For ease of presentation, we redefine the equilibrium based on the buying probability of the agents. We will show that they are equivalent. Its proof is in Appendix A.

Definition 2.2. *The probability vector* $\mathbf{q} = (q_1, q_2, ..., q_n) \in [0,1]^n$ *is an* equilibrium *at price* p*, if (where* med *is the median function)*

$$\forall i \in [n]\ , q_i = \mathsf{Pr}_{v_i \sim U(a_i,b_i)}\left[v_i - p + \sum_{j\in[n]} T_{j,i}\cdot q_j \geq 0\right]$$
$$= \text{med}\left\{0, 1, \frac{b_i - p + \sum_{j\in[n]} T_{j,i} q_j}{b_i - a_i}\right\}. \quad (2)$$

Lemma 2.3. *Given equilibrium* $\mathbf{q}$*, the strategy profile such agent* i *"buys the product if and only if her internal valuation* $v_i \geq p - \sum_{j\neq i} T_{j,i} q_j$*" is a Bayesian Nash equilibrium; on the contrary, if a strategy profile is a Bayesian Nash equilibrium, then the probability that agent* i *buys the product satisfies Equation 2.*

Equation 2 can be also defined in the language of a transfer function, which we will extensively reply on in the rest of the paper.

Definition 2.4 (Transfer function)**.** *Given price* p*, we define the transfer function* $f_p : [0,1]^n \to [0,1]^n$ *as*

$$[f_p(\mathbf{q})]_i = \text{med}\{0, 1, [g_p(\mathbf{q})]_i\} \quad (3)$$

in which

$$[g_p(\mathbf{q})]_i = \frac{b_i - p + \sum_{j\in[n]} T_{j,i} q_j}{b_i - a_i}.$$

Notice that $\mathbf{q}$ *is an equilibrium at price* p *if and only* $f_p(\mathbf{q}) = \mathbf{q}$.

Using Brouwer fixed point theorem, the existence of BNE is not surprising, even when influences are negative. However, we will show in Appendix C.1 that computing BNE will be PPAD-hard with negative influences. We now define the pessimistic and optimistic equilibria (similar to e.g. Van Zandt and Vives [VZV07]) based on the transfer function.

Definition 2.5. *Let* $f_p^{(1)} = f_p$*, and* $f_p^{(m)}(\mathbf{q}) = f_p(f_p^{(m-1)}(\mathbf{q}))$ *for* $m \geq 2$*. When all influences are* non-negative*, we define*

- **Pessimistic equilibrium***:* $\underline{\mathbf{q}}(p) = \lim_{m\to\infty} f_p^{(m)}(\mathbf{0})$*;*
- **Optimistic equilibrium***:* $\overline{\mathbf{q}}(p) = \lim_{m\to\infty} f_p^{(m)}(\mathbf{1})$.

We remark that both limits exist by monotonicity of f (see Fact 2.6 below), when all influences are non-negative. In addition, $\underline{\mathbf{q}}(p)$ and $\overline{\mathbf{q}}(p)$ are both equilibria themselves, because $f_p(\underline{\mathbf{q}}(p)) = \underline{\mathbf{q}}(p)$ and $f_p(\overline{\mathbf{q}}(p)) = \overline{\mathbf{q}}(p)$. We later show that $\underline{\mathbf{q}}(p)$ and $\overline{\mathbf{q}}(p)$ are the lower bound and upper bound for any equilibrium at price p respectively. Now we state some properties of equilibria, which we will use extensively later. Their proofs are in Appendix A.

For two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, we write $\mathbf{v}_1 \geq \mathbf{v}_2$ if $\forall i \in [n]$, $[\mathbf{v}_1]_i \geq [\mathbf{v}_2]_i$ and we write $\mathbf{v}_1 > \mathbf{v}_2$ if $\mathbf{v}_1 \geq \mathbf{v}_2 \wedge \mathbf{v}_1 \neq \mathbf{v}_2$.

Fact 2.6. *When all influences are non-negative, given* $p_1 \le p_2, \mathbf{q}^1 \le \mathbf{q}^2$*, the transfer function satisfies* $f_{p_2}(\mathbf{q}^1) \le f_{p_1}(\mathbf{q}^1) \le f_{p_1}(\mathbf{q}^2)$.

Lemma 2.7. *When all influences are non-negative, equilibria satisfy the following properties:*

a) *For any equilibrium* $\mathbf{q}$ *at price* p*, we have* $\underline{\mathbf{q}}(p) \le \mathbf{q} \le \overline{\mathbf{q}}(p)$.

b) *Given price* p*, for any probability vector* $\mathbf{q} \le \underline{\mathbf{q}}(p)$*, we have* $f_p^{(\infty)}(\mathbf{0}) = \underline{\mathbf{q}}(p) = f_p^{(\infty)}(\mathbf{q})$.

c) *Given price* $p_1 \le p_2$*, we have* $\underline{\mathbf{q}}(p_1) \ge \underline{\mathbf{q}}(p_2)$ *and* $\overline{\mathbf{q}}(p_1) \ge \overline{\mathbf{q}}(p_2)$.

d) $\underline{\mathbf{q}}(p) = \lim_{\varepsilon \to 0+} \underline{\mathbf{q}}(p+\varepsilon)$ *and* $\overline{\mathbf{q}}(p) = \lim_{\varepsilon \to 0-} \underline{\mathbf{q}}(p+\varepsilon)$.

In this paper, we consider the problem that whether we can exactly calculate the pessimistic (resp. optimistic) equilibrium, and whether we can maximize the revenue. The latter is formally defined as follows:

Definition 2.8 (Revenue maximization problem)**.**
Assume the value of agent i *is sampled from* $U(a_i, b_i)$ *and the influence matrix* T *is given. The revenue maximization problem is to compute an optimal price with respect to the* pessimistic equilibrium *(resp.* optimistic equilibrium *):*

$$\arg\max_{p>0} \sum_{i\in[n]} p \cdot [\underline{\mathbf{q}}(p)]_i \quad \textit{(resp. } \arg\max_{p>0} \sum_{i\in[n]} p \cdot [\overline{\mathbf{q}}(p)]_i \textit{).}$$

Notice that the optimal revenue with respect to the pessimistic equilibrium is robust against equilibrium selection. By Lemma 2.7(a), no matter which equilibrium the agents choose, this revenue is a minimal guarantee from the seller's perspective. The revenue guarantees for pessimistic and optimistic equilibria is an important objective to study; see for instance the *price of anarchy* and the *price of stability* in [NRTV07] for details.

3 The Main Algorithm

When all influences are non-negative, can we calculate $\underline{\mathbf{q}}(p)$ and $\overline{\mathbf{q}}(p)$ in polynomial time? We answer this question positively in this section by providing an efficient algorithm which computes the optimal revenue as well as the $\underline{\mathbf{q}}(p)$ and $\overline{\mathbf{q}}(p)$ for any price p.

3.1 A counter example for iterative method

Before coming to our efficient algorithm, notice that it is possible to iteratively apply the transfer function (Equation 3) to reach the equilibria by definition. However, this may require exponential number of steps to converge, as illustrated in the following example.

$$\begin{cases} p = 1 \\ [a_1, b_1] = [0, 2], \quad [a_i, b_i] = [0, 1] (2 \le i \le n) \\ \mathrm{T}_{i,i+1} = 0.5 (1 \le i \le n-2), \quad \mathrm{T}_{n-1,n} = \mathrm{T}_{n,n-1} = 1, \quad \text{other } \mathrm{T}_{j,i} = 0 \end{cases}$$

we can obtain that

$$f_p^{(n-2)}(\mathbf{0}) = (1/2, 1/2^2, ..., 1/2^{n-2}, 0, 0)$$

We proceed the calculation:

$$\begin{cases} f_p^{(n-2+2k)}(\mathbf{0}) & = (1/2, 1/2^2, ..., 1/2^{n-2}, k/2^{n-1}, k/2^{n-1}), \quad \text{if} \quad 0 \le k \le 2^{n-2} \\ f_p^{(n-2+2k+1)}(\mathbf{0}) & = (1/2, 1/2^2, ..., 1/2^{n-2}, (k+1)/2^{n-1}, k/2^{n-1}), \quad \text{if} \quad 0 \le k < 2^{n-2} \\ f_p^{(\infty)}(\mathbf{0}) & = (1/2, 1/2^2, ..., 1/2^{n-2}, 1, 1) \end{cases}$$

It can be seen from above that it takes $\Omega(2^n)$ number of steps before we reach the fixed point.

3.2 Outline of our line sweep algorithm

We start to introduce our algorithm with the easy case where valuations of agents are fixed. Consider the pessimistic decision vector $\underline{\mathbf{d}}(p)$ as a function of p. By monotonicity, there are at most $O(n)$ different such vectors when p varies from $+\infty$ to 0. In particular, at each price p, if we decrease p gradually to some threshold value, one more agent would change his decision to buy the product. Naturally, such kind of process can be casted in the "line sweep algorithm" paradigm.

When the private valuations of the agents are sampled from uniform distributions, the line sweep algorithm is much more complicated. We now introduce the algorithm to obtain the pessimistic equilibrium $\underline{\mathbf{q}}(p)$, while the method to obtain $\overline{\mathbf{q}}(p)$ is similar.[1] The essence of the line sweep algorithm is processing the events corresponding to some structural changes. We define the possible structures of a probability vector as follows.

Definition 3.1. *Given $\mathbf{q} \in [0,1]^n$, we define the structure function $S : [0,1]^n \to \{0, \star, 1\}^n$ satisfying:*

$$[S(\mathbf{q})]_i = \begin{cases} 0, & q_i = 0 \\ \star, & q_i \in (0,1) \\ 1, & q_i = 1. \end{cases} \tag{4}$$

Our line sweep algorithm is based on the following fact: when p is sufficiently large, obviously $\underline{\mathbf{q}}(p) = \mathbf{0}$; with the decreasing of p, at some point $p = p_1$ the pessimistic equilibrium $\underline{\mathbf{q}}(p)$ becomes non-zero, and there exists some *structural change* at this moment. Due to the monotonicity of $\underline{\mathbf{q}}(p)$ in Lemma 2.7, such structural changes can happen at most $2n$ times. (Each agent i can contribute to at most two changes: $0 \to \star$ and $\star \to 1$.) Therefore, there exist threshold prices $p_1 > p_2 > \cdots > p_m$ for $m \le 2n$ such that within two consecutive prices, the structure of the pessimistic equilibrium remains unchanged and $\underline{\mathbf{q}}(p)$ is a linear function of p. This indicates that the total revenue, i.e., $p \cdot \sum_i [\underline{\mathbf{q}}(p)]_i$, and its maximum value is easy to obtain. If we can compute the threshold prices and the corresponding pessimistic equilibrium $\underline{\mathbf{q}}(p)$ as a function of p, it will be straightforward to determine the optimal price p.

There are several difficulties to address in this line sweep algorithm.

- First, degeneracies, i.e., more than one structural changes in one event, are intrinsic in our problem. Unlike geometric problems where degeneracies can often be eliminated by perturbations, the degeneracies in our problem are persistent to small perturbations.

- Second, to deal with degeneracies, we need to identify the next structural change, which is related to the eigenvector corresponding to the largest eigenvalue of a linear operator. By a careful inspection, we avoid solving eigen systems so that our algorithm can be implemented by pure algebraic computations.

[1] We sweep the price from $+\infty$ to 0 to compute the pessimistic equilibrium, but we need to sweep from 0 to $+\infty$ for the optimistic one.

- Third, after the next change is identified, the usual method of pushing the sweeping line further does not work directly in our case. Instead, we recursively solve a subproblem and combine the solution of the subproblem with the current one to a global solution. The polynomial complexity of our algorithm is guaranteed by the monotonicity of the structures.

We first design a line sweep algorithm for the problem with a diagonal dominant condition, which will not contain degenerate cases, in Section 3.3. Then we describe techniques to deal with the unrestricted case in Section 3.4.

3.3 Diagonal dominant case

Definition 3.2 (Diagonal dominant condition)**.**
Let $L_{i,j} = T_{j,i}/(b_i - a_i)$ *and* $L_{i,i} = \mathrm{T}_{i,i} = 0$. *The matrix* $I - L$ *is* strictly diagonal dominant, *if* $\sum_j L_{i,j} = \sum_j \mathrm{T}_{j,i}/(b_i - a_i) < 1$.

This condition has some natural interpretation on the buying behavior of the agents. It means that the decision of any agent cannot be solely determined by the decisions of her friends. In particular, the following two situations cannot occur *simultaneously* for any agent i and price p: a) agent i will not buy the product regardless of her own valuation when none of her friends bought the product($p \geq b_i$), and b) agent i will always buy the product regardless of her own valuation when all her friends bought the product ($\sum_j T_{j,i} + a_i \geq p$).

In our line sweep algorithm, we maintain a *partition* $Z \cup W \cup O = V = [n]$, and name Z the *zero set*, W the *working set* and O the *one set*. This corresponds to the structure $\mathbf{s} \in \{0, \star, 1\}^n$ as follows:

$$s_i = 0\,(\forall i \in Z), \quad s_i = \star\,(\forall i \in W), \quad s_i = 1\,(\forall i \in O).$$

We use $\mathbf{x}_W$ or $[\mathbf{x}]_W$ to denote the restriction of vector $\mathbf{x}$ on set W, and for simplicity we write $\langle \mathbf{x}_Z, \mathbf{x}_W, \mathbf{x}_O \rangle = \mathbf{x}$. Let $L_{W\times W}$ be the projection of matrix L to $W \times W$, and $f|_W$ be the restriction of function f on W.

We start from the price $p = +\infty$ where the structure of the pessimistic equilibrium $\underline{\mathbf{q}}(p)$ is $\mathbf{s}^0 = \mathbf{0}$, i.e., $Z = [n]$ and $W = O = \emptyset$. The first event happens when p drops to $p_1 = \max_i b_i$ and $\underline{\mathbf{q}}(p)$ starts to become non-zero.

Assume now we have reached threshold price p_t, the current pessimistic equilibrium is $\mathbf{q}^t = \underline{\mathbf{q}}(p_t)$, and the structure in interval (p_t, p_{t-1}) (or $(p_t, +\infty)$ if $t = 1$) is $\mathbf{s}^{t-1}$. We define

$$\mathbf{x} = \left(\frac{b_1 - p_t}{b_1 - a_1}, \frac{b_2 - p_t}{b_2 - a_2}, \ldots, \frac{b_n - p_t}{b_n - a_n}\right)^T, \text{ and } \mathbf{y} = \left(\frac{1}{b_1 - a_1}, \frac{1}{b_2 - a_2}, \cdots, \frac{1}{b_n - a_n}\right)^T.$$

To analyze the pessimistic equilibrium in the next price interval, for price $p = p_t - \varepsilon$ where $\varepsilon > 0$, we write function $g_p(\cdot)$ (recall Equation 3) as:

$$g_{p_t - \varepsilon}(\mathbf{q}) = \mathbf{x} + \varepsilon \mathbf{y} + L\mathbf{q}.$$

For $p \in (p_t, p_{t-1})$, let partition $Z \cup W \cup O = [n]$ be consistent with the structure $\mathbf{s}^{t-1}$. According to Def. 3.1 and the right continuity $\mathbf{q}^t = \lim_{p \to p_t+} \underline{\mathbf{q}}(p)$ (see Lemma 2.7d), we have

$$\begin{array}{ll} \forall i \in Z, & [g_{p_t}(\mathbf{q}^t)]_i = [\mathbf{x} + L\mathbf{q}^t]_i \leq 0 \\ \forall i \in W, & [g_{p_t}(\mathbf{q}^t)]_i = [\mathbf{x} + L\mathbf{q}^t]_i \in (0, 1] \\ \forall i \in O, & [g_{p_t}(\mathbf{q}^t)]_i = [\mathbf{x} + L\mathbf{q}^t]_i \geq 1 \end{array} \tag{5}$$

Step 1: *For any $i \in Z$, if $[\mathbf{x} + L\mathbf{q}^t]_i = 0$, move i from zero set Z to working set W; for any $i \in W$, if $[\mathbf{x} + L\mathbf{q}^t]_i = 1$, move i from working set W to one set O.*

Notice that the structural changes we apply in Step 1 are exactly the changes defining the threshold price p_t. We will see in a moment that after the process in Step 1, the new partition will be the next structure $\mathbf{s}^t$ for $p \in (p_{t+1}, p_t)$. In other words, there is no more structural change at price p_t.

In the next two steps, we calculate the next threshold price p_{t+1}. For notation simplicity, we assume Z, W and O remain unchanged in these two steps. When p decreases by ε, we show that the probability vector of agents in W, $[\underline{\mathbf{q}}(p)]_W$, increases linearly with respect to ε. (See $\mathbf{r}_W(\varepsilon)$ below.) However, this linearity holds until we reach some point, where the next structural change takes place.

Step 2: *Define the vector $\mathbf{r}(\varepsilon) \in \mathbb{R}^n$, and let:*

$$\begin{aligned}
\mathbf{r}_W(\varepsilon) \quad &= \varepsilon(I - L_{W\times W})^{-1}\mathbf{y}_W + \mathbf{q}^t_W \\
&= \varepsilon(I - L_{W\times W})^{-1}\mathbf{y}_W + [\mathbf{x} + L\mathbf{q}^t]_W \\
\mathbf{r}_Z(\varepsilon) \quad &= \mathbf{x}_Z + \varepsilon\mathbf{y}_Z + L_{Z\times W}\mathbf{r}_W(\varepsilon) + L_{Z\times O}\mathbf{1}_O \\
&= \varepsilon(\mathbf{y}_Z + L_{Z\times W}(I - L_{W\times W})^{-1}\mathbf{y}_W) + [\mathbf{x} + L\mathbf{q}^t]_Z \\
\mathbf{r}_O(\varepsilon) \quad &= \mathbf{x}_O + \varepsilon\mathbf{y}_O + L_{O\times W}\mathbf{r}_W(\varepsilon) + L_{O\times O}\mathbf{1}_O \\
&= \varepsilon(\mathbf{y}_O + L_{O\times W}(I - L_{W\times W})^{-1}\mathbf{y}_W) + [\mathbf{x} + L\mathbf{q}^t]_O
\end{aligned} \tag{6}$$

Clearly, $\mathbf{r}(\varepsilon)$ is linear to ε and we write $\mathbf{r}(\varepsilon) = \varepsilon\boldsymbol{\ell} + (\mathbf{x} + L\mathbf{q}^t)$ where $\boldsymbol{\ell} = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in \mathbb{R}^n$ is the linear coefficient derived from Equation 6. When $I - L$ is strictly diagonal dominant, the largest eigenvalue of $L_{W\times W}$ is smaller than 1. Using this property one can verify (see Lemma B.1) that $\boldsymbol{\ell}$ is strictly positive.

Step 3:

$$\varepsilon_{min} = \min\left\{\min_{i\in Z}\left\{\frac{0 - [\mathbf{x} + L\mathbf{q}^t]_i}{\ell_i}\right\}, \min_{i\in W}\left\{\frac{1 - [\mathbf{x} + L\mathbf{q}^t]_i}{\ell_i}\right\}\right\} \tag{7}$$

Using the positiveness of vector $\boldsymbol{\ell}$ one can verify that $\varepsilon_{min} > 0$ (see Lemma B.1). We show that the next threshold price $p_{t+1} = p_t - \varepsilon_{min}$ by the following lemma. The proof is in the Appendix B.

Lemma 3.3. *$\forall 0 < \varepsilon \le \varepsilon_{min}$, $\underline{\mathbf{q}}(p_t - \varepsilon) = \langle \mathbf{0}_Z, \mathbf{r}_W(\varepsilon), \mathbf{1}_O \rangle$.*

We remark here that the above lemma has confirmed that our structural adjustments in Step 1 are correct and complete. Now we let $p_{t+1} = p_t - \varepsilon_{min}, \mathbf{q}^{t+1} = \langle \mathbf{0}_Z, \mathbf{r}_W(\varepsilon_{min}), \mathbf{1}_O \rangle$. The next structural change will take place at $p = p_{t+1}$. This is because according to the definition of ε_{min} (Equation 7), there must be some

$$i \in W \wedge [\mathbf{x} + \varepsilon_{min}\mathbf{y} + L\mathbf{q}^{t+1}]_i = 1, \quad \text{or} \quad i \in Z \wedge [\mathbf{x} + \varepsilon_{min}\mathbf{y} + L\mathbf{q}^{t+1}]_i = 0.$$

One can see that in the next iteration, this i will move to one set O or working set W accordingly. Therefore, we can iteratively execute the above three steps by sweeping the price further down. For completeness, we attach the pseudocode in Algorithm 1 in Appendix B.

The return value of our constrained line sweep method is a function $\underline{\mathbf{q}}$ which gives the pessimistic equilibrium for any price $p \in \mathbb{R}$, and $\underline{\mathbf{q}}(p)$ is a piecewise linear function of p with no more than $2n + 1$ pieces. All three steps in our algorithm can be done in polynomial time. Since there are only $O(n)$ threshold prices, we have the following result.

Theorem 3.4. *When the matrix $I - L$ is* strictly diagonal dominant*, Algorithm 1 calculates the pessimistic equilibrium $\underline{\mathbf{q}}(p)$ (resp. $\overline{\mathbf{q}}(p)$) for any given price p in polynomial time, together with the optimal revenue.*

3.4 General case

After relaxing the diagonal dominance condition, the algorithm becomes more complicated. This can be seen from this simple scenario. There are 2 agents, with $[a_1, b_1] = [a_2, b_2] = [0, 1]$, and $\mathrm{T}_{1,2} = \mathrm{T}_{2,1} = 2$. One can verify that $\underline{\mathbf{q}}(p) = (0, 0)^T$ when $p \geq 1$; $\underline{\mathbf{q}}(p) = (1, 1)^T$ when $p < 1$.

In this example, there is an *equilibrium jump* at price $p = 1$, i.e., $\underline{\mathbf{q}}(1) \neq \lim_{p \to 1-} \underline{\mathbf{q}}(p)$. Algorithm 1 essentially requires that both the left and the right continuity of $\underline{\mathbf{q}}(p)$. However, only the right continuity is unconditional by Lemma 2.7d. More importantly, degeneracies may occur: the new structure $\mathbf{s}^t$ when $p = p_t$ cannot be determined all in once in Step 1. When p goes from $p_t + \varepsilon$ to $p_t - \varepsilon$, there might take place even two-stage jumps: some index i might leave Z for O, without being in the intermediate state.

Let $\rho(L)$ be the largest norm of the eigenvalues in matrix L. The ultimate reason for such degeneracies, is $\rho(L_{W \times W}) \geq 1$ and $(I - L_{W \times W})^{-1} \neq \lim_{m \to \infty} (I + L_{W \times W} + \cdots + L_{W \times W}^{m-1})$. We will prove shortly in such cases, those structural changes in Step 1 are *incomplete*, that is, as p sweeps across p_t, at least one more structural change will take place. We derive a method to identify one *pivot*, i.e. an additional structural change, in polynomial time. Afterwards, we recursively solve a subproblem with set O taken out, and combine the solution from the subproblem with the current one. The follow lemma shows that whether $\rho(L) < 1$ can be determined efficiently.

Lemma 3.5. *Given non-negative matrix M, if $I - M$ is reversible and $(I - M)^{-1}$ is also non-negative, then $\rho(M) < 1$; on the contrary, if $I - M$ is degenerate or if $(I - M)^{-1}$ contains negative entries, $\rho(M) \geq 1$.*

3.4.1 Finding the pivot.

When $\rho(L_{W \times W}) < 1$ for the new working set W, one can find the next threshold price p_{t+1} following Step 2 and 3 in the previous subsection. Now, we deal with the case that $\rho(L_{W \times W}) \geq 1$ by showing that there must exists some additional agent $i \in W$ such that $[\underline{\mathbf{q}}(p)]_i = 1$ for any p smaller than the current price. We call such agent a *pivot*.

Since $\rho(L_{W \times W}) \geq 1$, we can always find a non-empty set $W_1 \subset W$ and $W_2 = W_1 \cup \{w\} \subset W$, satisfying $\rho(L_{W_1 \times W_1}) < 1$ but $\rho(L_{W_2 \times W_2}) \geq 1$. The pair (W_1, W_2) can be found by ordering the elements in W and add them to W_1 one by one. We now show that there is a pivot in W_2.

As $L_{W_2 \times W_2}$ is a non-negative matrix, based on Lemma B.2 there exists a non-zero eigenvector $\mathbf{u}_{W_2} \geq \mathbf{0}_{W_2}$ such that $L_{W_2 \times W_2} \mathbf{u}_{W_2} = \lambda \mathbf{u}_{W_2}$ and $\lambda = \rho(L_{W_2 \times W_2}) \geq 1$. $\mathbf{u}_{W_2}$ can be extended to $[n]$ by defining $\mathbf{u}_{[n] \setminus W_2} = \mathbf{0}_{[n] \setminus W_2}$. Let

$$k = \operatorname*{arg\,min}_{k \in W_2, u_k \neq 0} \frac{1 - q_k^t}{u_k} = \operatorname*{arg\,min}_{k \in [n], u_k \neq 0} \frac{1 - q_k^t}{u_k} \tag{8}$$

Now we prove that k is a pivot. Intuitively, if we slightly increase the probability vector $\mathbf{q}_{W_2}^t$ by $\delta \mathbf{u}_{W_2}$, where δ is a small constant, by performing the transfer function only on agents in W m times, their probability will increase by $\delta(1 + \lambda + .. + \lambda^m)\mathbf{u}_{W_2}$, while $\lambda \geq 1$. Therefore, after performing the transfer function sufficiently many times, agent $k \in W_2$'s probability will hit 1 first.

Lemma 3.6. *$\forall W_2 \subset W$ s.t. $\rho(L_{W_2\times W_2}) \geq 1$, we have $\forall \varepsilon > 0$, $[\underline{\mathbf{q}}(p_t - \varepsilon)]_k = 1$.*

We remark that if we can exactly estimate the eigenvector (which may be irrational), then the above lemma has already determined that the k defined in Equation 8 is a pivot. To avoid the eigenvalue computation, we find a quasi-eigenvector $\mathbf{u}$ in the following manner.

$$\mathbf{u} = \begin{cases} \mathbf{u}_{W_1} = (I - L_{W_1\times W_1})^{-1} L_{W_1\times\{w\}}; \\ u_w = 1; \\ \mathbf{u}_{Z\cup O\cup W\setminus W_2} = \mathbf{0}_{Z\cup O\cup W\setminus W_2}. \end{cases} \tag{9}$$

The meaning of the above vector is as follows. If we raise agent w's probability by δ, those probabilities of agents in W_1 increase proportionally to $L_{W_1\times\{w\}}\delta$. Assuming that we ignore the probability changes outside W_2 (which will even increase the probabilities in W_2), the probability of agents in W_1 will eventually converge to $(I + L_{W_1\times W_1} + L^2_{W_1\times W_1} + ...)L_{W_1\times\{w\}}\delta = (I - L_{W_1\times W_1})^{-1}L_{W_1\times\{w\}}\delta$.

We will see that the real probability vector increases at least "as much as if we increase in the direction of $\mathbf{u}$". In other words, we pick a pivot in the same way as Equation 8. The following is the critical lemma to support our result.

Lemma 3.7. *Given the definition of $\mathbf{u}$ in Equation 9 and k using Equation 8, we have $\forall \varepsilon > 0, [\underline{\mathbf{q}}(p_t - \varepsilon)]_k = 1$.*

3.4.2 Recursion on the subproblem.

Let $W' = W \setminus \{k\}$, $O' = O \cup \{k\}$, and we consider a subproblem with $n' = n - |O'| < n$ agents, where k is the pivot identified in the previous section. This subproblem is a projection of the original one, assuming that the agents in O' always tend to buy the product.

$$\forall i \in Z \cup W', \quad [a'_i, b'_i] = [a_i + \textstyle\sum_{j\in O'} T_{j,i}, b_i + \sum_{j\in O'} T_{j,i}]. \tag{10}$$

By recursively solving this new instance, we can solve the pessimistic equilibrium of the subproblem for any given price p. This recursive procedure will eventually terminate because every invocation reduces the number of agents by at least 1. The following lemma tells us that for any $p < p_t$, the pessimistic equilibrium of the original problem and the subproblem are one-to-one.

Lemma 3.8. *Let $\underline{\mathbf{q}}'(p)$ be the pessimistic equilibrium function in the subproblem. We have:*

$$\forall p < p_t, \underline{\mathbf{q}}(p) = \langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle.$$

At this moment we have solved the pessimistic equilibrium $\underline{\mathbf{q}}(p)$ for $p < p_t$, and thus solved the original problem. We summarize our unrestricted line sweep method in Algorithm 2 in Appendix B for completeness. Again $\underline{\mathbf{q}}(p)$ is a piecewise linear function of p with no more than $2n + 1$ pieces.

Theorem 3.9. *For matrix T satisfying $T_{i,i} = 0$ and $T_{i,j} \geq 0$, in polynomial time Algorithm 2 is able to calculate the pessimistic equilibrium $\underline{\mathbf{q}}(p)$ (resp. $\overline{\mathbf{q}}(p)$) at any price p, along with the optimal p that ensures the maximal revenue under the pessimistic equilibrium (resp. the optimistic equilibrium).*

4 Extensions

We discuss some possible extensions of our model in this section with both positive and negative influences. When the influence values can be negative, it is actually PPAD-hard to compute an *approximate* equilibrium. We define a probability vector $\mathbf{q}$ to be an ε-approximate equilibrium for price p if:

$$q_i \in (q'_i - \varepsilon, q'_i + \varepsilon),$$

where $q'_i = \text{med}\{0, 1, \frac{b_i - p + \sum_{j\in[n]} T_{j,i} q_j}{b_i - a_i}\}$. We have the following theorem, whose proof is deferred to Appendix C.1.

Theorem 4.1. *It is PPAD-hard to compute an n^{-c}-approximate equilibrium of our pricing system for any $c > 1$ when influences can be negative.*

In discriminative pricing setting, we study the revenue maximization problem in two natural models. We assume the agents are partitioned into k groups. The seller can offer different prices to different groups. The first model we consider is the fixed partition model, i.e., the partition is predefined. In the second model, we allow the seller to partition the agents into k groups and offer prices to the groups respectively. We have the following two theorems, whose proofs are deferred to Appendix C.2.

Theorem 4.2. *There is an FPTAS for the discriminative pricing problem in the fixed partition case with constant k.*

Theorem 4.3. *It is NP-hard to compute the optimal pessimistic discriminative pricing equilibrium in the choosing partition case.*

Appendix

A Missing Proofs in Section 2

Before proving Lemma 2.3, let us recall the *Bayesian Nash Equilibrium* (BNE) from the economics literature (see e.g. Chapter 8 of [MCWG95]). Formally, in a Bayesian game, each agent has a private type $v_i \in \Omega_i$, where Ω_i is the set of all possible types of agent i. Let $\mathcal{S}_i$ be the action space for agent i. Slightly abusing the notation, we define the (pure) strategy of agent i as a function $d_i : \Omega_i \to \mathcal{S}_i$. The utility of agent i when the type configuration $\mathbf{v}$ is known is $\mathcal{U}_i(\langle d_i(v_i), \mathbf{d}_{-i}(\mathbf{v}_{-i})\rangle, v_i)$, where $\mathbf{d}_{-i}(\mathbf{v}_{-i})$ is the joint actions of all agents other than i. Define the expected utility of agent i as:

$$\tilde{\mathcal{U}}_i(d_1(\cdot), \dots, d_n(\cdot)) := \mathbb{E}_{\mathbf{v}\sim\Omega_1\times\dots\times\Omega_n}[\mathcal{U}_i(\langle d_i(v_i), \mathbf{d}_{-i}(\mathbf{v}_{-i})\rangle, v_i)],$$

where the expectation is taking over all type configurations of the agents.

Definition A.1 (Bayesian Nash Equilibrium (**BNE**))**.** *A profile of strategies $\{d_1(\cdot), \dots, d_n(\cdot)\}$ is a (pure) Bayesian Nash Equilibrium, if and only if, for all i, all $v_i \in \Omega_i$ and any other strategy $d'_i(\cdot)$ of agent i, such that,*

$$\tilde{\mathcal{U}}_i(d_1(\cdot), \dots, d_i(\cdot), \dots, d_n(\cdot)) \ge \tilde{\mathcal{U}}_i(d_1(\cdot), \dots, d'_i(\cdot), \dots, d_n(\cdot))$$

In our setting, Ω_i is the set of private values of agent i and $d_i(\cdot)$ maps a particular value v_i to $\{0, 1\}$. The utility of agent i is given in Equation 1. Notice that mixed strategies are almost irrelevant here, because while fixing other agent's private valuations, agent i's strategy is a simple choice between to buy or not to buy. Unless the utility function $u_i(S, p) = 0$, there is always a unique better choice for her.

For ease of presentation, we redefine the equilibrium based on the buying probability of the agents and show that they are equivalent.

Lemma 2.3 (restated). *Given equilibrium $\mathbf{q}$ (recall Def. 2.2), the strategy profile such agent i "buys the product if and only if her internal valuation $v_i \ge p - \sum_{j\neq i} T_{j,i} q_j$" is a Bayesian Nash equilibrium; on the contrary, if a strategy profile is a Bayesian Nash equilibrium, then the probability that agent i buys the product satisfies Equation 2.*

Proof. Let strategy profile $\mathbf{d}(\cdot) = (d_1(\cdot), d_2(\cdot), ..., d_n(\cdot))$ be a Bayesian Nash equilibrium, and $q_i = \mathsf{Pr}_{v_i}[d_i(v_i) = 1]$ be the probability that agent i buys the product under this profile. In our setting, the utility of agent i is defined by Equation 1. Now we calculate the expected utility of agent i:

$$\begin{aligned} \tilde{u}_i(d_i(\cdot), \mathbf{d}_{-i}(\cdot)) &= \mathbb{E}_{v_i}[d_i(v_i) \cdot (v_i - p + \mathbb{E}_{\mathbf{v}_{-i}}[\sum_{j\neq i} \mathrm{T}_{j,i} d_j(v_j)])] \\ &= \mathbb{E}_{v_i}[d_i(v_i) \cdot (v_i - p + \sum_{j\neq i} \mathrm{T}_{j,i} q_j)] \end{aligned} \tag{11}$$

To satisfy the condition of Bayesian Nash equilibrium, we must have that $\forall d'_i(\cdot)$, $\tilde{u}_i(d_i(\cdot), \mathbf{d}_{-i}(\cdot)) \ge \tilde{u}_i(d'_i(\cdot), \mathbf{d}_{-i}(\cdot))$. This means, $d_i(v_i)$ must be 1 whenever $v_i - p + \sum_{j\neq i} \mathrm{T}_{j,i} q_j$ is positive, and 0 whenever it is negative [2]. Therefore, $q_i = \mathsf{Pr}[d_i(v_i) = 1] = \mathsf{Pr}[v_i - p + \sum_{j\neq i} \mathrm{T}_{j,i} q_j > 0]$, satisfying Def. 2.2.

[2] Strictly speaking, we should say "almost everywhere" but this does not affect our analysis.

On the contrary, the strategy that agent i "buys whenever $v_i \geq p - \sum_{j \neq i} \mathrm{T}_{j,i} q_j$" can be denoted as $d_i(v_i) = \mathbb{I}[v_i - p + \sum_{j \neq i} \mathrm{T}_{j,i} q_j > 0]$ where $\mathbb{I}$ is the indicator function. This obviously maximizes Equation 11, and is a Bayesian Nash equilibrium. □

Lemma 2.7 (restated). *Equilibria satisfy the following properties:*

a) *For any equilibrium* $\mathbf{q}$ *at price* p*, we have* $\underline{\mathbf{q}}(p) \leq \mathbf{q} \leq \overline{\mathbf{q}}(p)$.

b) *Given price* p*, for any probability vector* $\mathbf{q} \leq \underline{\mathbf{q}}(p)$*, we have* $f_p^{(\infty)}(\mathbf{0}) = \underline{\mathbf{q}}(p) = f_p^{(\infty)}(\mathbf{q})$.

c) *Given price* $p_1 \leq p_2$*, we have* $\underline{\mathbf{q}}(p_1) \geq \underline{\mathbf{q}}(p_2)$ *and* $\overline{\mathbf{q}}(p_1) \geq \overline{\mathbf{q}}(p_2)$.

d) $\underline{\mathbf{q}}(p) = \lim_{\varepsilon \to 0+} \underline{\mathbf{q}}(p + \varepsilon)$ *and* $\overline{\mathbf{q}}(p) = \lim_{\varepsilon \to 0-} \underline{\mathbf{q}}(p + \varepsilon)$.

Proof.

a) By the definition of equilibrium, $\mathbf{q} = f_p(\mathbf{q}) = f_p^{(\infty)}(\mathbf{q})$. Next according to $\mathbf{0} \leq \mathbf{q} \leq \mathbf{1}$ and the monotonicity of f_p, we derive that:

$$f_p(\mathbf{0}) \leq f_p(\mathbf{q}) \leq f_p(\mathbf{1}) \Rightarrow \ldots \Rightarrow f_p^{(\infty)}(\mathbf{0}) \leq f_p^{(\infty)}(\mathbf{q}) \leq f_p^{(\infty)}(\mathbf{1}).$$

b) By symmetry we only need to prove the first half. We already know that $f_p(\underline{\mathbf{q}}(p)) = \underline{\mathbf{q}}(p)$, then recall the monotonicity of f_p

$$\begin{aligned} \mathbf{0} \leq \mathbf{q} \leq \underline{\mathbf{q}}(p) &\Rightarrow f_p(\mathbf{0}) \leq f_p(\mathbf{q}) \leq f_p(\mathbf{q}(p)) \Rightarrow \ldots \\ &\Rightarrow f_p^{(\infty)}(\mathbf{0}) \leq f_p^{(\infty)}(\mathbf{q}) \leq f_p^{(\infty)}(\underline{\mathbf{q}}(p)) \\ &\Rightarrow \underline{\mathbf{q}}(p) \leq f_p^{(\infty)}(\mathbf{q}) \leq \underline{\mathbf{q}}(p). \end{aligned}$$

Notice that the last "$\Rightarrow$" is due to $f_p^{(\infty)}(\mathbf{0}) = \underline{\mathbf{q}}(p) = f_p(\underline{\mathbf{q}}(p)) = \ldots = f_p^{(\infty)}(\underline{\mathbf{q}}(p))$, while the convergence of the limit $f_p^{(\infty)}(\mathbf{q}) = \lim_{m \to \infty} f_p^{(\infty)}(\mathbf{q})$ is ensured by the sandwich theorem.

c) This time we use the combined monotonicity of the function f (Fact 2.6)

$$\begin{aligned} p_1 \leq p_2 \wedge \mathbf{0} \geq \mathbf{0} &\Rightarrow f_{p_1}(\mathbf{0}) \geq f_{p_2}(\mathbf{0}) \\ p_1 \leq p_2 \wedge f_{p_1}(\mathbf{0}) \geq f_{p_2}(\mathbf{0}) &\Rightarrow f_{p_1}^{(2)} \geq f_{p_2}^{(2)}(\mathbf{0}) \\ &\ldots \\ &\Rightarrow f_{p_1}^{(\infty)}(\mathbf{0}) \geq f_{p_2}^{(\infty)}(\mathbf{0}) \\ &\Rightarrow \underline{\mathbf{q}}(p_1) \geq \underline{\mathbf{q}}(p_2) \end{aligned}$$

For similar reason we also have $\overline{\mathbf{q}}(p_1) \geq \overline{\mathbf{q}}(p_2)$.

d) We only prove the first half while the property of $\overline{\mathbf{q}}(p)$ can be obtained in similar way. We first claim that for any fixed m, $f_p^{(m)}(\mathbf{0}) = \lim_{\varepsilon \to 0+} f_{p+\varepsilon}^{(m)}(\mathbf{0})$. Since $f_p(\mathbf{q})$ is a continuous multi-variable function with respect to $(p, \mathbf{q})$, the composition $f_p^{(m)}(\mathbf{q})$ is also continuous. This directly implies our claim.

Now assume Property (d) is not true: there exists $\delta > 0$ and ε_0 such that $\forall 0 < \varepsilon < \varepsilon_0$, $[\underline{\mathbf{q}}(p) - \underline{\mathbf{q}}(p+\varepsilon)]_i > \delta$ for some i. By definition of the pessimistic equilibrium, there exists m_0 such that $[\underline{\mathbf{q}}(p) - f_p^{(m_0)}(\mathbf{0})]_i < \delta/2$. On the other hand by our claim just proved, we can choose ε small enough such that $[f_p^{(m_0)}(\mathbf{0}) - f_{p+\varepsilon}^{(m_0)}(\mathbf{0})]_i < \delta/2$. Combining the two we have $\delta > [\underline{\mathbf{q}}(p) - f_{p+\varepsilon}^{(m_0)}(\mathbf{0})]_i \geq [\underline{\mathbf{q}}(p) - \underline{\mathbf{q}}(p+\varepsilon)]_i$, where the second inequality is due to non-decreasing sequence $\{f_{p+\varepsilon}^{(m)}(\mathbf{0})\}_{m\geq 1}$ that converge to $\underline{\mathbf{q}}(p+\varepsilon)$. This contradiction completes the proof. We remark here that the left continuity does not hold, see the beginning of Section 3.4.

□

B Missing Proofs in Section 3

Before proving Lemma 3.3, we first show Equation 7 is well defined.

Lemma B.1. *$\boldsymbol{\ell} \in \mathbb{R}_+^n$ and $\varepsilon_{min} > 0$.*

Proof. When $I - L$ is strictly diagonal dominant, the largest eigenvalue of $L_{W\times W}$ is smaller than 1. By the knowledge from complex analysis, the following limit exists

$$(I - L_{W\times W})^{-1} = I + L_{W\times W} + L_{W\times W}^2 + \cdots$$

and it is a non-negative matrix since L is non-negative.

Now, $\mathbf{y}$ is strictly positive and therefore $\boldsymbol{\ell}_W = (I - L_{W\times W})^{-1}\mathbf{y}_W \in \mathbb{R}_+^{|W|}$ is also positive. Besides, recall the definition in Equation 6 we have $\boldsymbol{\ell}_Z = \mathbf{y}_Z + L_{Z\times W}\boldsymbol{\ell}_W \in \mathbb{R}_+^{|Z|}$, $\boldsymbol{\ell}_O = \mathbf{y}_O + L_{O\times W}\boldsymbol{\ell}_W \in \mathbb{R}_+^{|W|}$, and therefore $\boldsymbol{\ell} \in \mathbb{R}_+^n$. Finally, by our Step 1, we have $[\mathbf{x} + L\mathbf{q}^t]_i < 0$ for $i \in Z$, and $[\mathbf{x} + L\mathbf{q}^t]_j < 1$ for $j \in W$. Therefore, $\varepsilon_{min} > 0$ is properly defined. □

Lemma 3.3 (restated). $\forall 0 < \varepsilon \leq \varepsilon_{min}$, $\underline{\mathbf{q}}(p_t - \varepsilon) = \langle \mathbf{0}_Z, \mathbf{r}_W(\varepsilon), \mathbf{1}_O \rangle$.

Proof. We first show that $\mathbf{q} = \langle \mathbf{0}_Z, \mathbf{r}_W(\varepsilon), \mathbf{1}_O \rangle$ is an equilibrium for $\varepsilon \in (0, \varepsilon_{min}]$. By our definition of ε_{min}, when $0 < \varepsilon \leq \varepsilon_{min}$ we must have

$$\begin{cases} [g_{p_t-\varepsilon}(\mathbf{q})]_W = \mathbf{r}_W(\varepsilon) \in [0,1]^{|W|} \\ [g_{p_t-\varepsilon}(\mathbf{q})]_Z = \mathbf{r}_Z(\varepsilon) \leq \mathbf{0}_Z \\ [g_{p_t-\varepsilon}(\mathbf{q})]_W = \mathbf{r}_O(\varepsilon) \geq \mathbf{r}_O(0) \geq \mathbf{1}_O \end{cases}$$

Since $f_{p_t-\varepsilon} = \mathrm{med}\{0, 1, g_{p_t-\varepsilon}\}$ (Def. 2.4), it must be the case that $f_{p_t-\varepsilon}(\mathbf{q}) = \mathbf{q}$, i.e., $\mathbf{q}$ is an equilibrium. Next we lower bound the pessimistic equilibrium by $\underline{\mathbf{q}}(p_t - \varepsilon) \geq \mathbf{q}$. This will be sufficient to complete the proof following from Lemma 2.7a.

Denote $p = p_t - \varepsilon$, notice that $\underline{\mathbf{q}}(p) = f_p^{(\infty)}(\mathbf{0}) = f_p^{(\infty)}(\mathbf{q}^t)$, where the second equality is because:

$$\text{Lemma 2.7c} \Rightarrow \mathbf{q}^t = \underline{\mathbf{q}}(p_t) \leq \underline{\mathbf{q}}(p)$$
$$\stackrel{\text{Lemma 2.7b}}{\Longrightarrow} \underline{\mathbf{q}}(p) = f_p^{(\infty)}(\mathbf{0}) = f_p^{(\infty)}(\mathbf{q}^t)$$

For the simplicity of notation, we define $\mathbf{x}'_W := \mathbf{x}_W + L_{W\times O}\mathbf{1}_O$ as a constant vector, and according to the definition of an equilibrium:

$$\mathbf{q}_W^t = \mathbf{x}'_W + L_{W\times W}\mathbf{q}_W^t.$$

After repeated use of the monotonicity of transfer function f, we make the following analysis [3] :

$$
\begin{cases}
f_{p_t-\varepsilon}(\mathbf{q}^t) \geq \langle \mathbf{0}_Z, \varepsilon \mathbf{y}_W + \mathbf{q}^t_W, \mathbf{1}_O \rangle \\
f^{(2)}_{p_t-\varepsilon}(\mathbf{q}^t) \geq f_{p_t-\varepsilon}(\langle \mathbf{0}_Z, \varepsilon \mathbf{y}_W + \mathbf{q}^t_W, \mathbf{1}_O \rangle) \\
\qquad\qquad \geq \langle \mathbf{0}_Z, \mathbf{x}'_W + \varepsilon \mathbf{y}_W + L_{W\times W}(\varepsilon \mathbf{y}_W + \mathbf{q}^t_W), \mathbf{1}_O \rangle \\
\qquad\qquad = \langle \mathbf{0}_Z, \varepsilon (I + L_{W\times W}) \mathbf{y}_W + \mathbf{q}^t_W, \mathbf{1}_O \rangle \\
\qquad\qquad \dots \\
f^{(\infty)}_{p_t-\varepsilon}(\mathbf{q}^t) \geq \langle \mathbf{0}_Z, \varepsilon (\sum_{i=0}^{\infty} \left(L_{W\times W} \right)^i \Big) \mathbf{y}_W + \mathbf{q}^t_W, \mathbf{1}_O \rangle
\end{cases}
\tag{12}
$$

The last inequality in Equation 12 implies that

$$
\begin{aligned}
\underline{\mathbf{q}}(p_t - \varepsilon) &= f^{(\infty)}_{p_t-\varepsilon}(\mathbf{q}^t) \geq \langle \mathbf{0}_Z, \varepsilon (I - L_{W\times W})^{-1} \mathbf{y}_W + \mathbf{q}^t_W, \mathbf{1}_O \rangle \\
&= \langle \mathbf{0}_Z, \mathbf{r}_W(\varepsilon), \mathbf{1}_O \rangle = \mathbf{q}
\end{aligned}
$$

□

The following lemma in matrix analysis is important for our analysis.

Lemma B.2 ([HJ90]). *Given a non-negative matrix M (i.e. $\forall i, j, M_{ij} \geq 0$), there exists a non-negative (and non-zero) eigenvector $\mathbf{x} \geq \mathbf{0}$ satisfying $M\mathbf{x} = \lambda \mathbf{x}$, in which $\lambda = \rho(M)$ is a real number.*

Lemma 3.5 (restated). *Given non-negative matrix M, if $I - M$ is reversible and $(I - M)^{-1}$ is also non-negative, then $\rho(M) < 1$; on the contrary, if $I - M$ is degenerate or if $(I - M)^{-1}$ contains negative entry, $\rho(M) \geq 1$.*

Proof. For the first half, assume the contrary that $\rho(M) \geq 1$. According to Lemma B.2, there exists a non-negative $\mathbf{x}$ s.t. $(I - M)\mathbf{x} = (1 - \rho(M))\mathbf{x} \leq \mathbf{0}$. As $(I - M)^{-1}$ is non-negative, multiply a non-positive vector $(I - M)\mathbf{x}$ to its right is also non-positive: $(I - M)^{-1}(I - M)\mathbf{x} \leq \mathbf{0}$. But the last inequality means that $x \leq \mathbf{0}$ which contradicts the result in Lemma B.2 saying x is a non-negative and non-zero eigenvector.

For the second half, if $I - M$ is degenerate then $(I - M)\mathbf{x} = 0$ has a non-zero solution, which implies $M\mathbf{x} = \mathbf{x}$ and $\rho(M) \geq 1$. Assume the contrary that $\rho(M) < 1$, then $(I - M)^{-1} = I + M + M^2 + ...$ is non-negative as M is non-negative, resulting in a contradiction. □

Lemma 3.6 (restated). *For any $W_2 \subset W$ s.t. $\rho(L_{W_2\times W_2}) \geq 1$, we have $\forall \varepsilon > 0$, $[\underline{\mathbf{q}}(p_t - \varepsilon)]_k = 1$.*

Proof. We will only prove the statement when $W_2 = W$, as the analysis for $W_2 \neq W$ is similar.

[3] within which we implicitly adopted the following property:

$$
\varepsilon(I + L_{W\times W} + ... + L^{m-1}_{W\times W})\mathbf{y}_W + \mathbf{q}^t_W \leq \varepsilon_{min}(I - L_{W\times W})^{-1}\mathbf{y}_W + \mathbf{q}^t_W \leq \mathbf{1}_W
$$

Algorithm 1 ConstrainedLineSweepMethod$(n, T, \mathbf{a}, \mathbf{b})$

Input: $n, T, \mathbf{a}, \mathbf{b}$.

Output: The pessimistic equilibrium function $\underline{\mathbf{q}} : p \mapsto \underline{\mathbf{q}}(p)$.

$L_{i,j} \leftarrow \mathrm{T}_{j,i}/(b_i - a_i)$;
$p_1 \leftarrow \max_{1 \leq i \leq n} b_i$;
$\underline{\mathbf{q}}(p)|_{[p_1,\infty)} \leftarrow \mathbf{0}$;
$Z \leftarrow [n]$; $W \leftarrow \emptyset$; $O \leftarrow \emptyset$; $t \leftarrow 1$;
while $\underline{\mathbf{q}}(p_t) \neq \mathbf{1}$ **do**
    $\mathbf{q}^t \leftarrow \underline{\mathbf{q}}(p_t)$;
    $\mathbf{x} \leftarrow ((b_1 - p_t)/(b_1 - a_1), (b_2 - p_t)/(b_2 - a_2), ..., (b_n - p_t)/(b_n - a_n))$;
    $\mathbf{y} \leftarrow (1/(b_1 - a_1), 1/(b_2 - a_2), ..., 1/(b_n - a_n))^T$;
    for all $i \in Z$ s.t. $\mathbf{x}_i + \sum_j L_{i,j} q_j^t = 0$ **do**
        $Z \leftarrow Z \setminus \{i\}$; $W \leftarrow W \cup \{i\}$;
    end for
    for all $i \in W$ s.t. $\mathbf{x}_i + \sum_j L_{i,j} q_j^t = 1$ **do**
        $W \leftarrow W \setminus \{i\}$; $O \leftarrow O \cup \{i\}$;
    end for
    $\boldsymbol{\ell}_W \leftarrow (I - L_{W \times W})^{-1} \mathbf{y}_W$; {See Equation 6}
    $\boldsymbol{\ell}_Z \leftarrow \mathbf{y}_Z + L_{Z \times W} \boldsymbol{\ell}_W$; {See Equation 6}
    $\varepsilon_{min} = \min\{\min_{i \in Z}\{\frac{0 - [\mathbf{x} + L\mathbf{q}^t]_i}{\ell_i}\}, \min_{i \in W}\{\frac{1 - [\mathbf{x} + L\mathbf{q}^t]_i}{\ell_i}\}\}$; {See Equation 7}
    $p_{t+1} \leftarrow p_t - \varepsilon_{min}$;
    $\underline{\mathbf{q}}(p)|_{[p_{t+1}, p_t)} \leftarrow \langle \mathbf{1}_Z, \mathbf{r}_W(p_t - p), \mathbf{1}_O \rangle$;
    $t \leftarrow t + 1$;
end while
$\underline{\mathbf{q}}(p)|_{(-\infty, p_t)} \leftarrow \mathbf{1}$;
return $\underline{\mathbf{q}}$;

As $L_{W\times W}$ is a non-negative matrix, based on Lemma B.2 there exists a non-zero eigenvector $\mathbf{u}_W \geq \mathbf{0}_W$ such that $L_{W\times W}\mathbf{u}_W = \lambda \mathbf{u}_W$ and $\lambda = \rho(L_{W\times W}) \geq 1$. $\mathbf{u}_W$ can be extended to $[n]$ by defining $\mathbf{u}_{Z\cup O} = \mathbf{0}_{Z\cup O}$. Let

$$k = \arg \min_{k\in[n],u_k\neq 0} \frac{1-q_k^t}{u_k}$$

Since $\mathbf{u} \neq \mathbf{0}$, the above equation is well defined. As $q_i^t < 1$ for any $i \in W$ in the current configuration, we also have $\frac{1-q_k^t}{u_k} > 0$. The tie is broken arbitrarily.

Since $\mathbf{y} > 0$ and $\mathbf{u} \geq 0$, for any $\varepsilon > 0$, there exists $\delta > 0$ satisfying $\delta\mathbf{u} \leq \varepsilon\mathbf{y}$. Because δ can be arbitrary small, let $\delta = \left(\frac{1-q_k^t}{u_k}\right)/(1+\lambda+\cdots+\lambda^{m-1})$ in which m is sufficiently large to satisfy the above constraint. For any probability vector $\mathbf{q}$, we have $[f_{p_t-\varepsilon}(\mathbf{q})]_i = \mathrm{med}\{0,1,[\mathbf{x}+\varepsilon\mathbf{y}+L\mathbf{q}]_i\}$. Define function $[h(\mathbf{q})]_i = \mathrm{med}\{0,1,[\mathbf{x}+\delta\mathbf{u}+L\mathbf{q}]_i\}$. Clearly, $f_{p_t-\varepsilon}(\mathbf{q}) \geq h(\mathbf{q})$.

Starting from $\mathbf{q}^t \geq \mathbf{q}^t$, we continue to apply the left side by $f_{p_t-\varepsilon}$ and the right side by h, we derive the followings: [4]

$$\begin{aligned}
f_{p_t-\varepsilon}(\mathbf{q}^t) &\geq h(\mathbf{q}^t) \geq \langle \mathbf{0}_Z, \mathbf{x}'_W + \delta\mathbf{u}_W + L_{W\times W}\mathbf{q}_W, \mathbf{1}_O\rangle \\
&= \langle \mathbf{0}_Z, \delta\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle \\
f^{(2)}_{p_t-\varepsilon}(\mathbf{q}^t) &\geq h(\langle \mathbf{0}_Z, \delta\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle) \\
&\geq \langle \mathbf{0}_Z, \delta(I + L_{W\times W})\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle \\
&= \langle(\mathbf{0}_Z, \delta(1+\lambda)\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle \\
&\cdots \\
f^{(m)}_{p_t-\varepsilon}(\mathbf{q}^t) &\geq \langle \mathbf{0}_Z, \delta(1+\lambda+\ldots+\lambda^{m-1})\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle \\
&= \langle \mathbf{0}_Z, \left(\frac{1-q_k^t}{u_k}\right)\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle
\end{aligned}$$

From our selection of k, we know that

$$\left[\langle \mathbf{0}_Z, \Big(\frac{1-q_k^t}{u_k}\Big)\mathbf{u}_W + \mathbf{q}_W^t, \mathbf{1}_O\rangle\right]_k = 1$$

i.e., $\forall\varepsilon > 0$, we have $[\underline{\mathbf{q}}(p_t-\varepsilon)]_k \geq [f^{(m)}_{p_t-\varepsilon}(\mathbf{q}^t)]_k = 1$. This completes the proof of the existence of pivot k. □

Lemma 3.7 (restated). *Given the definition of* $\mathbf{u}$ *in Equation 9 and* k *using Equation 8, we have* $\forall\varepsilon > 0, [\underline{\mathbf{q}}(p_t-\varepsilon)]_k = 1$.

Proof. We only prove the case when $Z = O = W \setminus W_2 = \emptyset$, and will briefly describe how our proof can be extended to the general case. We use $\mathbf{q}_{-w}$ to denote $\mathbf{q}_{[n]\setminus\{w\}} = \mathbf{q}_{W_1}$.

Let $\delta = \min_{k\in[n],u_k\neq 0} \frac{1-q_k^t}{u_k} > 0$. We know that if we increase from $\mathbf{q}^t$ in the direction of $\mathbf{u}$, we can at most raise $\delta\mathbf{u}$ until agent k's probability hits 1. For a fixed $\varepsilon > 0$, let $\mathbf{q}' = \underline{\mathbf{q}}(p_t-\varepsilon)$ be the pessimistic equilibrium. To prove $q'_k = 1$ we consider two cases:

[4] Within which we implicitly adopted the following property: $\forall m_0 < m$,

$$\delta(1+\lambda+\ldots+\lambda^{m_0-1})\mathbf{u}_W + \mathbf{q}_W^t \leq \left(\frac{1-q_k^t}{u_k}\right)\mathbf{u}_W + \mathbf{q}_W^t \leq \mathbf{1}_W$$

- $q'_w \geq q^t_w + \delta$.

  This means that in the real scenario, agent w indeed increases her probability by at least δ. It can be verified that in this case, the rest of the agents in $W_1 = [n] \setminus \{w\}$ have to increase by at least $\delta \mathbf{u}_{-w}$. In other words, $\mathbf{q}' - \mathbf{q}^t \geq \delta \mathbf{u}$ which already implies $q'_k \geq 1$ by our definition of k and δ.

- $q'_w < q^t_w + \delta$.

  In this case, the actual final probability of w is small. Let $\delta' = q'_w - q^t_w < \delta$. We start from the inequality $\mathbf{q}^t + \langle \mathbf{0}_{-w}, \delta' \rangle \geq \mathbf{q}^t$. Let $\mathbf{z}_{-w} = L_{W_1 \times \{w\}}$. By applying the transfer function $f_{p_t - \varepsilon}$ to both sides and using the monotonicity,

  $$\mathbf{q}^t + \langle \varepsilon \mathbf{y}_{-w} + \delta' \mathbf{z}_{-w}, \sigma_1 \rangle = f_{p_t-\varepsilon}(\mathbf{q}^t + \langle \mathbf{0}_{W_1}, \delta' \rangle) \geq f_{p_t-\varepsilon}(\mathbf{q}^t)$$

  for some $\sigma_1 \geq 0$. Based on $q^t_w + \delta' \geq q'_w = [f^{(\infty)}_{p_t-\varepsilon}(\mathbf{q}^t)]_w \geq [f_{p_t-\varepsilon}(\mathbf{q}^t)]_w$, we always have $\mathbf{q}^t + \langle \varepsilon \mathbf{y}_{-w} + \delta' \mathbf{z}_{-w}, \delta' \rangle \geq f_{p_t-\varepsilon}(\mathbf{q}^t)$. By applying the transfer function again we have

  $$\mathbf{q}^t + \langle \varepsilon (I + L_{W_1 \times W_1}) \mathbf{y}_{-w} + \delta' (I + L_{W_1 \times W_1}) \mathbf{z}_{-w}, \sigma_2 \rangle \geq f^{(2)}_{p_t-\varepsilon}(\mathbf{q}^t).$$

  We continue to replace σ_2 by δ' and apply the transfer function. Doing this iteratively while assuming that ε is sufficiently small, we have:

  $$\mathbf{q}^t + \langle \varepsilon (I - L_{W_1 \times W_1})^{-1} \mathbf{y}_{-w} + \delta' (I - L_{W_1 \times W_1})^{-1} \mathbf{z}_{-w}, \delta' \rangle \geq f^{(\infty)}_{p_t-\varepsilon}(\mathbf{q}^t).$$

  Recall the definition of $\mathbf{u}$ we can rewrite the above equation as: $\mathbf{q}^t + \delta' \mathbf{u} + \langle \varepsilon (I - L_{W_1 \times W_1})^{-1} \mathbf{y}_{-w}, 0 \rangle \geq \mathbf{q}'$. Since $\delta' < \delta$ and $\mathbf{q}^t < \mathbf{1}$, we have $\mathbf{q}^t + \delta' \mathbf{u} < \mathbf{1}$. When ε is sufficiently small, we also have that the left hand side in the above equation is smaller than $\mathbf{1}$, and this proves that $\mathbf{q}' < \mathbf{1}$ when ε is small, which contradicts Lemma 3.6 which says that the pivot always exists.

We describe how we prove the general case where $Z, W \setminus W_2$ and O are not necessarily empty. Imagine a subproblem with only $|W_2|$ rational players, while for agent $i \in [n] \setminus W_1$, her probability is fixed to q^t_i, no matter how the price varies and other players behave. We can also define the transfer function and pessimistic equilibrium in this subproblem. Then, using the same argument as above, we can find one pivot k such that agent k's probability hits 1 in the subproblem, when $p < p_t$. It can be verified that in the original problem, this agent k will also buy with probability 1, since when releasing the constraints on agents in $[n] \setminus W_2$, the entire probability vector may only increase rather than decrease. □

Algorithm 2 LineSweepMethod$(n, T, \mathbf{a}, \mathbf{b})$

Input: $n, T, \mathbf{a}, \mathbf{b}$.

Output: The pessimistic equilibrium function $\underline{\mathbf{q}} : p \mapsto \underline{\mathbf{q}}(p)$.

$L_{i,j} \leftarrow \mathrm{T}_{j,i}/(b_i - a_i)$;
$p_1 \leftarrow \max_{1 \le i \le n} b_i$;
$\underline{\mathbf{q}}(p)|_{[p_1,\infty)} \leftarrow \mathbf{0}$;
$Z \leftarrow [n]$; $W \leftarrow \emptyset$; $O \leftarrow \emptyset$; $t \leftarrow 1$;
while $\underline{\mathbf{q}}(p_t) \neq \mathbf{1}$ **do**
  $\mathbf{q}^t \leftarrow \underline{\mathbf{q}}(p_t)$;
  $\mathbf{x} \leftarrow ((b_1 - p_t)/(b_1 - a_1), (b_2 - p_t)/(b_2 - a_2), ..., (b_n - p_t)/(b_n - a_n))$;
  $\mathbf{y} \leftarrow (1/(b_1 - a_1), 1/(b_2 - a_2), ..., 1/(b_n - a_n))^T$;
  for all $i \in Z$ s.t. $\mathbf{x}_i + \sum_j L_{i,j} q^t_j = 0$ **do**
    $Z \leftarrow Z \setminus \{i\}$; $W \leftarrow W \cup \{i\}$;
  end for
  for all $i \in W$ s.t. $\mathbf{x}_i + \sum_j L_{i,j} q^t_j = 1$ **do**
    $W \leftarrow W \setminus \{i\}$; $O \leftarrow O \cup \{i\}$;
  end for
  if $\rho(L_{W\times W}) < 1$ **then**
    $\boldsymbol{\ell}_W \leftarrow (I - L_{W\times W})^{-1}\mathbf{y}_W$; and $\boldsymbol{\ell}_Z \leftarrow \mathbf{y}_Z + L_{Z\times W}\boldsymbol{\ell}_W$; {See Equation 6}
    $\varepsilon_{min} = \min\{\min_{i\in Z}\{\frac{0-[\mathbf{x}+L\mathbf{q}^t]_i}{\ell_i}\}, \min_{i\in W}\{\frac{1-[\mathbf{x}+L\mathbf{q}^t]_i}{\ell_i}\}\}$; {See Equation 7}
    $p_{t+1} \leftarrow p_t - \varepsilon_{min}$;
    $\underline{\mathbf{q}}(p)|_{[p_{t+1},p_t)} \leftarrow \langle \mathbf{1}_Z, \mathbf{r}_W(p_t - p), \mathbf{1}_O\rangle$;
  else $\{|W| \ge 2\}$
    Assume $W = \{w_1, w_2, ...w_{|W|}\}$;
    for $i \leftarrow 2$ to $|W|$ **do**
      $W_1 \leftarrow \{w_1, ...w_{i-1}\}$; $W_2 \leftarrow \{w_1, ...w_i\}$;
      if $\rho(L_{W_2\times W_2}) \ge 1$ **then**
        $\mathbf{u}_{W_1} = (I - L_{W_1\times W_1})^{-1} L_{W_1\times\{w_i\}}$;
        $u_{w_i} = 1$; $\mathbf{u}_{[n]\setminus W_2} = \mathbf{0}_{[n]\setminus W_2}$; {See Equation 9}
        $k \leftarrow \mathrm{argmin}_{k\in[n], u_k\neq 0}\{(1 - [\mathbf{q}^t]_k)/u_k\}$; {See Equation 8}
        $O \leftarrow O \cup \{k\}$; $\bar{O} \leftarrow [n] \setminus O$;
        $\forall i \in \bar{O}, \quad [a'_i, b'_i] = [a_i + \sum_{j\in O} \mathrm{T}_{j,i}, b_i + \sum_{j\in O} \mathrm{T}_{j,i}]$; {See Equation 10}
        $\underline{\mathbf{q}}' \leftarrow$ LineSweepMethod$(|\bar{O}|, T_{\bar{O}\times\bar{O}}, \mathbf{a}', \mathbf{b}')$;
        $\underline{\mathbf{q}}(p)|_{(-\infty,p_t)} \leftarrow \langle \underline{\mathbf{q}}'(p), \mathbf{1}_O\rangle$;
        return $\underline{\mathbf{q}}$;
      end if
    end for {never reach here}
  end if
  $t \leftarrow t + 1$;
end while
$\underline{\mathbf{q}}(p)|_{(-\infty,p_t)} \leftarrow \mathbf{1}$;
return $\underline{\mathbf{q}}$;

Lemma 3.8 (restated). *Let $\underline{\mathbf{q}}'(p)$ be the pessimistic equilibrium function in the subproblem. We have:*

$$\forall p < p_t, \underline{\mathbf{q}}(p) = \langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle.$$

Proof. We prove the lemma in two steps. We will first show that $\langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle$ is an equilibrium at price p, and then lower bound the pessimistic equilibrium by $\langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle \le \underline{\mathbf{q}}(p)$. Combined with the property of equilibrium in Lemma 2.7a, it is enough to see that $\langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle$ is the pessimistic equilibrium of the original problem.

For convenience let $\overline{O}' = [n] \setminus O'$.

- Let $\mathbf{q} = \langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle$, and we are going to show $f_p(\mathbf{q}) = \mathbf{q}$. Based on the definition of $[a_i', b_i']$ in the subproblem, we already have that $[f_p(\mathbf{q})]_{\overline{O}'} = \mathbf{q}_{\overline{O}'} = \underline{\mathbf{q}}'(p)$. This is because $\forall i \in \overline{O}'$,

$$\begin{aligned}[f_p(\mathbf{q})]_i &= \operatorname{med}\left\{0, 1, \frac{b_i - p + \sum_{j\in[n]} \mathrm{T}_{j,i} q_j}{b_i - a_i}\right\} \\ &= \operatorname{med}\left\{0, 1, \frac{b_i' - p + \sum_{j\in\overline{O}'} \mathrm{T}_{j,i} q_j}{b_i' - a_i'}\right\} = q_i.\end{aligned}$$

  Therefore we only need to show that $[f_p(\mathbf{q})]_{O'} = \mathbf{1}_{O'}$. Assume the contradiction that $[f_p(\mathbf{q})]_{O'} \le \mathbf{1}_{O'}$ and $\exists i \in O'$ s.t. $[f_p(\mathbf{q})]_i < 1$. We start from $f_p(\mathbf{q}) \le \mathbf{q}$ and arrive at $f_p^{(m)}(\mathbf{q}) \le f_p^{(m-1)}(\mathbf{q})$ by using the monotonicity of f. The following limit exists because a non-increasing and lower bounded sequence has a limit.

$$\mathbf{q}^* = \lim_{m\to\infty} f_p^{(m)}(\mathbf{q}) \le f_p(\mathbf{q})$$

  Because of the continuity of function f, $\mathbf{q}^*$ is an equilibrium at price p. According to Lemma 2.7

$$[\underline{\mathbf{q}}(p)]_i \le q_i^* \le [f_p(\mathbf{q})]_i < 1.$$

  If $i \in O = O' \setminus \{k\}$, this contradict the fact that $1 = [\underline{\mathbf{q}}(p_t)]_i \le [\underline{\mathbf{q}}(p)]_i$; if $i = k$ this contradicts Lemma 3.7. Therefore it must be the case that $f_p(\mathbf{q}) = \mathbf{q}$.

- We now lower bound the pessimistic equilibrium $\underline{\mathbf{q}}(p)$. For similar reason as the first half of the proof, we have $[\underline{\mathbf{q}}(p)]_{O'} = \mathbf{1}_{O'}$. Let f_p' be the transfer function of the subproblem. We start from the inequality $\langle \mathbf{0}_{\overline{O}'}, \mathbf{1}_{O'} \rangle \le \underline{\mathbf{q}}(p)$ and apply the monotone function f_p to both sides:

$$f_p(\langle \mathbf{0}_{\overline{O}'}, \mathbf{1}_{O'} \rangle) = \langle f_p'(\mathbf{0}_{\overline{O}'}), \star \rangle \le \underline{\mathbf{q}}(p)$$

  We need not to know what $\star$ is, and start with the new inequality $\langle f_p'(\mathbf{0}_{\overline{O}'}), \mathbf{1}_{O'} \rangle \le \underline{\mathbf{q}}(p)$ and derive that:

$$f_p(\langle f_p'(\mathbf{0}_{\overline{O}'}), \mathbf{1}_{O'} \rangle) = \langle f_p'^{(2)}(\mathbf{0}_{\overline{O}'}), \star \rangle \le \underline{\mathbf{q}}(p)$$

  By doing this again and again, we reach the inequality

$$\langle f_p'^{(\infty)}(\mathbf{0}_{\overline{O}'}), \mathbf{1}_{O'} \rangle \le \underline{\mathbf{q}}(p)$$

  which immediately gives us $\langle \underline{\mathbf{q}}'(p), \mathbf{1}_{O'} \rangle \le \underline{\mathbf{q}}(p)$.

This completes the proof. □

C Missing Proofs in Section 4

C.1 Hardness results with negative influences

In this section, we show that when the influence values can be negative, it is PPAD-hard to compute an *approximate* equilibrium. We define a probability vector $\mathbf{q}$ to be an ε-approximate equilibrium for price p if:

$$q_i \in (q_i' - \varepsilon, q_i' + \varepsilon),$$

where $q_i' = \text{med}\{0, 1, \frac{b_i - p + \sum_{j \in [n]} T_{j,i} q_j}{b_i - a_i}\}$.

We prove the PPAD hardness by a reduction from the two player Nash equilibrium computation. Our construction is inspired by [CT11]. Let matrices $A, B \in \mathcal{R}^{n \times n}$ be the payoff matrices of the two players respectively, i.e. $(A_i)_j$ (resp. $(B_i)_j$) is the payoff for the first player (resp. the second player) when the first player plays its i-th strategy and the second player plays its j-th strategy. It is PPAD-hard to approximate the two player Nash Equilibrium with error $1/n^{\alpha}$ for any constant $\alpha > 0$ [CDT09]. We build an instance of our pricing problem as follows. (δ is a small value to be determined later.)

- Price $p = 1/2$.
- User X_i with value interval $[0, 1]$ for $i \in [n]$. The probability that X_i buys the product is x_i.
- User Y_i with value interval $[0, 1]$ for $i \in [n]$. The probability that Y_i buys the product is y_i.
- User $U_{i,j}$, $i, j \in [n]$ is used to enforce $x_i = 0$, when $A_i y^T + \delta < A_j y^T$. For any $k \in [n]$, we assign influence on edge $(Y_k, U_{i,j})$ to be $(A_j)_k - (A_i)_k$. Define $U_{i,j}$'s valuation interval to be $[1/2 - \delta, 1/2 - \delta + \delta^2]$.
- User $V_{i,j}$, $i, j \in [n]$ is used to enforce $y_i = 0$ when $B_i x^T + \delta < B_j x^T$. For any $k \in [n]$, influence on edge $(X_k, V_{i,j})$ is $(B_j)_k - (B_i)_k$. Define $V_{i,j}$'s valuation to be $[1/2 - \delta, 1/2 - \delta + \delta^2]$.
- For $i, j \in [n]$, influence values on edges $(U_{i,j}, X_i)$ and $(V_{i,j}, Y_i)$ are -1.
- All other pair-wise influence values are zero.

In our setting, if $U_{i,j}$ buys the product, it will provide influence of -1 to X_i, which will imply the probability that X_i will buy the product is 0.

Theorem 4.1 (restated). *It is PPAD-hard to compute an n^{-c}-approximate equilibrium of our pricing system for any $c > 1$ when influences can be negative.*

Proof. Let $\delta = n^{-c}$. Consider the instance we constructed above. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$ be the set of vectors that form an δ-approximate equilibrium of our pricing instance. We will show that we can construct an $O(n^{1-c})$ approximate Nash equilibrium for the two player game. To simply the notation, we define $x \pm y = [x - y, x + y]$. In particular, we have

$$\begin{aligned}
x_i &\in \text{med}\{0, 1, 1/2 - \sum\nolimits_i u_{i,j}\} \pm \delta \\
y_i &\in \text{med}\{0, 1, 1/2 - \sum\nolimits_i v_{i,j}\} \pm \delta \\
u_{i,j} &\in \text{med}\{0, 1, 1 - 1/\delta + 1/\delta^2 \langle A_j - A_i, \mathbf{y} \rangle\} \pm \delta \\
v_{i,j} &\in \text{med}\{0, 1, 1 - 1/\delta + 1/\delta^2 \langle B_j - B_i, \mathbf{y} \rangle\} \pm \delta
\end{aligned}$$

For the purpose of controlling normalization, we first prove that $||\mathbf{x}||_\infty = ||\mathbf{y}||_\infty \in 1/2 \pm \delta$. It is clear that $||\mathbf{x}||_\infty \le 1/2+\delta$, since X_i receives no positive influence in our construction. Furthermore, for any vector $\mathbf{y}$, let $t = \arg\max_{i\in[n]}\{A_i\mathbf{y}^T\}$. Then for each $U_{t,j}$, the sum of influence is $(A_j - A_t)\mathbf{y}^T \le 0$. As a result, $U_{t,j}$ will never buy the product and give a negative influence to X_t, which implies $||\mathbf{x}||_\infty \ge x_t \ge 1/2-\delta$. The proof of $||\mathbf{y}||_\infty \in 1/2 \pm \delta$ is similar. We can define

$$[\mathbf{x}']_i = \begin{cases} [\mathbf{x}]_i & \text{if } [\mathbf{x}]_i > \delta \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we obtain $\mathbf{y}'$. We then normalize them to $\mathbf{x}^* = \frac{\mathbf{x}'}{||\mathbf{x}'||_1}$ and $\mathbf{y}^* = \frac{\mathbf{y}'}{||\mathbf{y}'||_1}$. It is sufficient to prove that $\mathbf{x}^*$ and $\mathbf{y}^*$ form an $9n\delta$-approximate Nash for the two player game. In particular, we shall show

$$\langle A_i, \mathbf{y}^*\rangle + 6n\delta < \langle A_j, \mathbf{y}^*\rangle \Longrightarrow x_i^* = 0$$

$$\langle B_i, \mathbf{x}^*\rangle + 6n\delta < \langle B_j, \mathbf{x}^*\rangle \Longrightarrow y_i^* = 0$$

When $A_i\mathbf{y}^*+6n\delta < A_j\mathbf{y}^*$, clearly $\langle A_j - A_i, \mathbf{y}'\rangle > 6n\delta||\mathbf{y}'||_1 > 3n\delta$ and $\langle A_j - A_i, \mathbf{y}\rangle > 3n\delta - 2n\delta \ge n\delta$. (The entries in A and B are within range $[-1, 1]$.) Therefore, $u_{i,j} \ge 1-\delta$, which implies $x_i \le \delta$ and $x_i^* = 0$ by our construction. The proof for the statement of $\mathbf{y}^*$ is symmetric. □

Theorem 4.1 implies that computing an exact equilibrium in our pricing system is PPAD-hard, when the price is given and the influence could be negative.

C.2 Discriminative pricing model

In this section, we discuss the extension of our problem in the discriminative pricing model, in which different agents may be offered with different prices to the same good, and there are at most k different prices offered. We only consider non-negative influences in this section. Let G be a k-partition of agent set $[n]$ and g_i denote the group which agent i belongs to. Let $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ be the price vector corresponding to the k groups in the partition. Def. 2.2, Def. 2.4, and Def. 2.5 for a single price p can be straightforwardly extended to the case of price vector $\mathbf{p}$ with partition G, and we omit their re-definitions here. We define the revenue maximization problem under the discriminative pricing model as follows.

Definition C.1. *The revenue maximization problem is to compute an optimal price vector* $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ *w.r.t. the pessimistic equilibrium (resp. optimistic equilibrium):*

$$\arg\max_{\mathbf{p}\ge\mathbf{0}} \sum_{i\in[n]} p_{g_i} \cdot [\underline{\mathbf{q}}(\mathbf{p})]_i \quad \textit{(resp.} \quad \arg\max_{\mathbf{p}\ge\mathbf{0}} \sum_{i\in[n]} p_{g_i} \cdot [\overline{\mathbf{q}}(\mathbf{p})]_i \ \textit{)}.$$

Apparently, the *uniform pricing* case is a special case of discriminative model when $k = 1$. In this section, we discuss two different cases in this model: the fixed partition case and the choosing partition case. As the name suggests, in the fixed partition case, the partition of the agents are given. On the other hand, in the choosing partition case, the algorithm has the flexibility to choose the partition.

C.2.1 Fixed partition case with constant k.

In this case, we let the k-partition of G be fixed and known to our algorithm. This is natural in a modern market such as setting prices based on different regions or different user memberships.

Our algorithm for the uniform pricing model can be extended to some restricted cases in the fixed partition case. For instance, given a fixed price vector $\mathbf{p} = \langle p_1, p_2, \ldots, p_n \rangle$, we consider (a) all possible price vectors that are $\{\mathbf{p}+x\mathbf{1} \mid x \in \mathbb{R}\}$; or (b) all possible price vectors that are $\{\xi\mathbf{p} \mid \xi > 0\}$. These two cases capture certain scenarios in which the prices in different partitions either follow fixed ratios, e.g. by different tax ratio or income distribution, or have fixed differences, e.g. by transportation costs. In both cases, we can reduce the problem to a uniform price one, which can be solved by our proposed algorithm. We only present the algorithm for the first case and the proof for the second case is similar.

Claim C.2. *There is a refinement of Algorithm 2 for all possible price vectors that are $\{\mathbf{p}+x\mathbf{1} \mid x \in \mathbb{R}\}$.*

Proof. In order to compute revenue with respect to price vector $\mathbf{p}$, we refine our line sweep method. Let $p_t = \min_{i\in[k]} p_i$ be the minimum entry in $\mathbf{p}$ and Δ_i be $p_{g_i} - p_t$. We use $\underline{\mathbf{q}}$ to denote the equilibrium when agents i is offered price $p + \Delta_i$ and modify Algorithm 2 line 7 to

$$\mathbf{x} \leftarrow \left(\frac{b_1 - p_t - \Delta_1}{b_1 - a_1}, \frac{b_2 - p_t - \Delta_2}{b_2 - a_2}, ..., \frac{b_n - p_t - \Delta_n}{b_n - a_n}\right)$$

□

If the space expanded by the price vectors is not one dimensional, enumerating all structures like our proposed line sweep algorithm is generally impractical. (See an counter example in Appendix D.)

When there is no constrain on the possible prices, we design an FPTAS when k is a constant. We first estimate the optimal revenue which we can hope to achieve. In particular, for any group $i \in [k]$, we set the prices for all other groups to be 0. By our algorithm in Section 3, we can compute the maximum revenue from group i in this case as R_i. Clearly, the optimal revenue is at most $R = \sum_{i\in[k]} R_k$. We then design a discretization scheme based on R.

Let $\varepsilon \in (0, 1)$ be a constant. Define $p_{max} = R$ and $p_{min} = \varepsilon R/(2kn)$. Our algorithm works as follows:

1 Compute revenue r_i when price vector is

$$\mathbf{p}_i = \left(0, (1+\varepsilon)^{i_1} p_{min}, (1+\varepsilon)^{i_2} p_{min}, \ldots, (1+\varepsilon)^{i_k} p_{min}\right)$$

for all $0 \le i_1, i_2, \ldots, i_k = \lceil \log_{1+\varepsilon} 2kn/\varepsilon \rceil$.

2 Return $\mathbf{p}_i$ with the maximum calculated r_i.

Theorem 4.2 (restated). *There is an FPTAS for the discriminative pricing problem in the fixed partition case with constant k.*

Proof. The set of total prices for each group in the algorithm is $O(\log_{1+\varepsilon}(n/\varepsilon)) = O(\frac{\log(kn/\varepsilon)}{\varepsilon})$. Enumerating all possible prices takes time $O(\log^k(n/\varepsilon)/\varepsilon^k)$, which is polynomial when k is constant.

Assume $\mathbf{p}_{opt}$ be the optimal price vector with optimal revenue $R_{opt} \ge \max_i R_i \ge R/k$. Let $\mathbf{p}'$ be the price vector, which is obtained by rounding all prices $\mathbf{p}_{opt}$ down to the closest Steiner price. Now consider the error introduced by the rounding scheme. Notice that by monotonicity, this rounding will only increase the buying probability of each user. For all prices that are rounded to 0, the revenue from the users offered with those prices is at most $\varepsilon R/(2k) \leq \varepsilon R_{opt}/2$ with the optimal price vector. All other prices are decreased by at most a factor of $1 + \varepsilon/2$. The revenue collected from the agents offered with those prices in $\mathbf{p}'$ is at least $1+\varepsilon/2$ of that with $\mathbf{p}$. Therefore, in total, we receive a revenue of at least $(1 - \varepsilon/2)R_{opt}/(1 + \varepsilon/2) \geq (1 - \varepsilon)R_{opt}$. □

C.2.2 Choosing partition case with constant k.

Now we consider the case that the partition G can be chosen by our algorithm in order to maximize the seller's revenue. More precisely we define our problem as follows. Given the distribution of agents' values and their influence network, the problem is to compute the optimal k-partition of G together with an optimal price vector $\mathbf{p}$ to maximize the seller's revenue. We prove that when the revenue is measured based on the pessimistic equilibrium, this optimization problem is NP-hard even in the fixed valuation case ($a_i = b_i$ for each player i).

In particular, we consider the following special case of the problem: (i) k = 2, (ii) The valuation of the agents is deterministic, and (iii) the price can only be 0 or 1. For the case $k > 2$, we can add some dummy agents in our construction and force the optimal solution to get the optimal revenue in our construction for $k = 2$. We summarize the main result in the following theorem.

Theorem 4.3 (restated). *It is NP-hard to compute the optimal pessimistic discriminative pricing equilibrium in the choosing partition case.*

Proof. We use a reduction from the *Vertex Cover* problem. We show that using any polynomial algorithm for the pessimistic discriminative pricing problem in choose partition case, any instance of the *Vertex Cover* problem can be solved in polynomial time. In an instance of an *Vertex Cover* problem, given a graph $G = (V, E)$, we must specify whether a subset $S \subset V$ exists such that $|S| \leq K$ and $\forall u, v$ such that $(u, v) \in E$, we have $v \in S$ or $u \in S$.

Then we prove that, for each graph $G(V, E)$, the exists a network $G'(V', E')$ and agents' valuation so that, if r_{opt} is the optimal revenue in G', vertex number in minimum vertex cover of G is $|V'| - r_{opt}$. First we will show how to construct G' from G. V' is formed from the union of three parts, denoted by A, D and M. In the first set A, there is one vertex a_i with initial value 0 for each vertex v_i in G. The set D is used to represent the edges in original graph G. There is a vertex d_e for each edge in G. The initial values of all these vertices are 0. Let $e = (v_i, v_j)$ be an edge in G. There is one edge from a_i, one from a_j to d_e weighted 1. The edges is used to represent the cover action, if a_i or a_j buys the product, the d_e vertex will also buy the product. In addition, we also need construct $|D| \times |A|$ edges weighted $\frac{1}{|D|}$ in G', which from each d_e to each a_i. The edges means only if we cover all the vertex in D, all the vertex in A will just reach the value 1. Finally, we must use a considerable large set $M(\geq |V|^3)$ to force the optimal solution to set an price on 1. This is because if no final price is 1, it is hard to guarantee all the vertex in D will buy the product, which also represent all the edges in G be covered. Therefore, we put independent vertexes in M and weight them with 1. It is obvious that we must set a price 1 and another 0 to get the best revenue and activate the vertexes in A and D.

Now we will show the optimal revenue r_{opt} in G' is equal to $|V'| - |C|$. Let C be minimum vertex cover of G and F be the set of people who get a zero price in G'. In our configuration, a final price of optimal solution must be 1, so the revenue is equal to $|V' - F|$. Firstly, we show that $r_{opt} \geq |V| - |C|$. If given a minimum vertex cover C for G, we can define F according to C. It means the seller will give the free product to the vertex in A if it represent a vertex in minimum cover C. By the definition of vertex covering, all the vertexes in D , which represents the all edges in original graph G, will be activated and their value will all reach to 1. As a result, the value of all vertexes in A will also reach to 1 and will buy the product. Conclusively, our revenue will reach to $|V'| - |C|$. At last, we will prove $r_{opt} \leq |V'| - |C|$. Suppose $r_{opt} > |V'| - |C|$, there must be a free set F to achieve the maximum revenue r_{opt}. Consider the structure of F, if $F \cap M$ is not empty, we can eliminate these vertexes to get a better revenue. If $F \cap D$ is not empty, each point $d \in F \cap D$ can be replaced by the vertex in A which have an edge to it. This replacement never decreases our revenue because the new vertex have a 1-weight edge to the old vertex. So there must be a $F \subseteq A$, which could make the revenue greater than $|V'| - |C|$. By the construction, we can convert F to a vertex cover in G. This is a contradiction to the definition of minimum vertex cover. □

D Counter Example in Appendix C.2

Assume n is even. Let p_1 be the price offered to agents $\{1, 3, \ldots, n-1\}$ and p_2 be the price offered to $\{2, 4, \ldots, n\}$. The influences are defined as $\mathrm{T}_{j,i} = 2^{\lceil j/2-1 \rceil}$ for $i < j$ and $j - i$ is odd and greater than 0, and 0 otherwise. The valuation of agent i is $2^{\lceil i/2-1 \rceil}$. There are a total of $2^{\Omega(n)}$ structures for the pessimistic equilibrium as p_1 and p_2 vary in $[0, +\infty)$.

Proof. We prove the following stronger statement by induction: for all prices $p_1 \in (0, 2^{n/2})$ and $p_2 \in (0, 2^{n/2})$, there are at least $2^{n/2}$ structures.

Consider the base case of $n = 2$, with agent 1 and 2. Since there is no influence among them, the number of structure configuration is certainly $4 > 2$, with the price range $p_1 \in (0, 2)$, $p_2 \in (0, 2)$. Suppose the statement is true for $n = 2i$. For the case of $n = 2(i + 1)$, there are two additional agents $2i + 1$ and $2i + 2$.

Consider the price range $p_1 \in (2^i, 2^{i+1})$, $p_2 \in (0, 2^i)$. Agent $2i+1$ will not buy the product while agent $2i+2$ does in this case. Since the influence from agent $2i+2$ to every agent with price p_1 is 2^i except $2i + 1$, the "effective price" for all odd agents except $2i + 1$ is $p_1 - 2^i \in (0, 2^i)$. In such price range, there are at least 2^i structures by induction. Symmetrically, the same conclusion holds for price range $p_1 \in (0, 2^i)$, $p_2 \in (2^i, 2^{i+1})$. Notice these two price ranges have difference configuration for agents $2i + 1$ and $2i + 2$. Therefore, in total there are at least $2 \cdot 2^i = 2^{i+1}$ structures. □

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