Asymptotically Optimal Strategy-Proof Mechanisms for Two-Facility Games

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ABSTRACT

We consider the problem of locating facilities in a metric space to serve a set of selfish agents. The cost of an agent is the distance between her own location and the nearest facility. The social cost is the total cost of the agents. We are interested in designing strategy-proof mechanisms *without payment* that have a small approximation ratio for social cost. A mechanism is a (possibly randomized) algorithm which maps the locations reported by the agents to the locations of the facilities. A mechanism is *strategy-proof* if no agent can benefit from misreporting her location in any configuration.

This setting was first studied by Procaccia and Tennenholtz [21]. They focused on the facility game where agents and facilities are located on the real line. Alon et al. studied the mechanisms for the facility games in a general metric space [1]. However, they focused on the games with only one facility. In this paper, we study the two-facility game in a general metric space, which extends both previous models.

We first prove an $\Omega(n)$ lower bound of the social cost approximation ratio for deterministic strategy-proof mechanisms. Our lower bound even holds for the line metric space. This significantly improves the previous *constant* lower bounds [21, 17]. Notice that there is a matching linear upper bound in the line metric space [21]. Next, we provide the first *randomized* strategy-proof mechanism with a constant approximation ratio of 4. Our mechanism works in general metric spaces. For randomized strategy-proof mechanisms, the previous best upper bound is $O(n)$ which works only in the line metric space.

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1. INTRODUCTION

We start with a typical problem in economics: the government plans to build several libraries in a city to serve a local community. All residents report their home addresses so that the government can decide the most appropriate library locations. Every resident wants to be as close to one of the libraries as possible; meanwhile, the government wants to minimize the sum of distances between each resident and her nearest library, which is called the *social cost*. In many cases, the government cannot trust the self-reported addresses from residents, because people are *selfish*, and could report false addresses for personal benefits.

This type of problem is called the *facility game*. In this game, *agents* report their locations and accordingly a *mechanism* chooses positions to build facilities. A mechanism is also called a *social choice* function in the Economics literature. Specifically, agents and facilities are located in some metric space. To model real problems, the distance function could be the Euclidean distance, the shortest path distance (in a graph), or any other metric. An agent may misreport her location if she can reduce her own cost. To avoid such misreport, the *strategy-proofness* is introduced in game theory, which is the main focus of this paper. In a strategy-proof mechanism, no agent can unilaterally benefit from misreporting. A stronger requirement is called *group strategy-proofness*. In a group strategy-proof mechanism, no group of agents can misreport their locations such that each member can strictly benefit. Formal definitions of the these concepts are given in Section 2.

The facility game has a rich history in social science literatures. There has been some partial characterizations of the strategy-proof mechanisms for some metric spaces, e.g.

a facility on a line [6, 19, 4, 25] or on a general network [23]. However, these works have not considered the optimizations or approximations over the social cost.

The study of algorithmic aspect of mechanism design problem was initiated by the seminal work of Nisan and Ronen [20] in 1999. During the past decade, a significant body of work has been done for optimization problems from a mechanism design point of view [15, 2, 10, 14]. Most of the work deals with mechanisms which employ payment. In particular, the well known Vickrey-Clarke-Groves (VCG) mechanism [26, 8, 12] is strategy-proof, which gives an optimal solution for our facility game if payment is allowed.

However, in many social choice settings, monetary transfer may be unavailable due to legal or ethical issues as noted by Schummer and Vohra [24]. Voting is one perfect example. More recently, Procaccia and Tennenholtz formally initiated the study of approximate mechanism design *without money* in their seminal paper [21]. This type of work can also be traced back to the work on incentive compatible learning by Dekel et al. [9]. From a more algorithmic viewpoint, Procaccia and Tennenholtz studied strategy-proof mechanisms that give provable approximation ratios on social cost. A mechanism is called γ-approximate, if for every input instance, the social cost for the outcome is no more than γ times that of an optimal assignment. We are interested in studying both upper and lower bounds of the approximation ratios for possibly randomized strategy-proof mechanisms. We note that here the lower bound is due to the cost of strategy-proofness rather than the computational complexity. Same type of lower bounds were proved for mechanisms (with payment) for scheduling unrelated machines [7, 13, 18, 16].

For the two-facility game on a line, Procaccia and Tennenholtz [21] gave an upper bound of $n-2$ and a lower bound of 1.5 for deterministic strategy-proof mechanisms. The lower bound was later improved to 2 [17]. In addition, Lu, Wang and Zhou [17] obtained an upper bound of $n/2$ and a lower bound of 1.045 for randomized strategy-proof mechanisms. To close the huge gaps for both deterministic and randomized cases is an important open problem in this direction. Our work resolves this problem by proving asymptotically tight bounds for both cases.

Besides, Alon et al. [1] studied the facility game in a general metric space rather than a line. They gave an almost complete characterization of the feasible strategy-proof approximation ratios, but under the condition that there is solely one facility. In this paper, we analyze the game with two facilities, and prove our results in any general metric space. Notice that this generalization is non-trivial and our work is a joint extension of the work by Procaccia and Tennenholtz [21] and the work by Alon et al. [1].

1.1 Our Results

We study the approximation ratios of strategy-proof mechanisms for two-facility games in *generalmetric spaces*. It is the first time that facility games with more than one facility are considered in general metric spaces. We obtain three main results.

Our first result is a linear lower bound of the approximation ratio for *deterministic* strategy-proof mechanisms. This is noticeably the first super constant lower bound for the two-facility game, and even holds in the line metric space. It confirms one conjecture in [21]. Moreover, the proof idea is new, and we highlight two key concepts we employ and may be of independent interest.

- *Partial group strategy-proofness.* In a partial group strategy-proof mechanism, a group of agents at the *same location* cannot benefit even if they misreport their locations simultaneously. As noted in [21], there is a lower bound of $\Omega(n)$ for group strategy-proof mechanisms of the two-facility game. However, a strategy-proof mechanism may not be group strategy-proof. To overcome such obstacle, we introduce the concept of partial group strategy-proofness and prove that it can be implied from the strategy-proofness. Our lower bound is benefited from this observation.

- *Image set* [1]. This is defined as the set of possible facility locations when a group of agents varies their reported locations within the entire space, fixing the locations of other agents. This concept allows an investigation of infinite number of location profiles simultaneously, while previous lower bounds are obtained by analyzing only constant many profiles.

We remark here that the above two concepts are defined for general facility games in an arbitrary metric space.

Our second result is a *randomized* strategy-proof mechanism with a *constant* approximation ratio for the two-facility game, working in general metric spaces. In comparison, the previous best known upper bound is $O(n)$ and works only in the line metric space. Together with our first result, this mechanism indicates that randomness is indeed an essential power in (money-less) strategy-proof mechanism design. This new mechanism is very intuitive. The first facility is allocated uniformly over all reported locations; the second facility is assigned to another reported location with probability proportional to its distance to the first facility. We call it the *Proportional Mechanism.* Although the mechanism seems natural, the proof of its strategy-proofness and the analysis of its approximation ratio are both involved.

Our third result is a deterministic mechanism with an $O(n)$ approximation ratio for the circle metric space. A circle is $S^1 \subset \mathbb{R}^2$, and the distance of two points on S^1 is the length of the minor arc between them. This is noticeably the first bounded deterministic mechanism for two-facility games over metric spaces other than the line. It is also worth pointing out that this mechanism is group strategy-proof.

We summarize our results and the state of the art in the following table.

	Deterministic	Randomized
Line	UB : $(n-2$ [21]$)$ LB : $\mathbf{\frac{n-1}{2}}$ (2 [17])	UB : $\mathbf{4}$ $(\frac{n}{2}$ [17]$)$ LB : (1.045 [17])
Circle	UB : $\mathbf{n-1}$ (N/A) LB : $\mathbf{\frac{n-1}{2}}$ (2 [17])	UB : $\mathbf{4}$ (N/A) LB : (1.045 [17])
General	UB : N/A LB : $\mathbf{\frac{n-1}{2}}$ (2 [17])	UB : $\mathbf{4}$ (N/A) LB : (1.045 [17])

Table 1: Our results are in bold. The expressions in brackets are previous results (N/A means no previous known bound).

We recall that even for the line metric space, the previous best upper and lower bounds are $O(n)$ and $\Omega(1)$ respectively,

[1]An anonymous reviewer pointed out that a similar idea is used in [5].

in both deterministic and randomized settings. This work significantly improves our understanding of: 1) the power of (money-less) strategy-proof mechanism for facility games; 2) the power of randomness in (money-less) strategy-proof mechanism design.

1.2 Related Work

The facility game problem has a rich history in social science literatures. Consider the case that we are building one facility in a discrete set of locations (alternatives). Agents are reporting their preferences for the alternatives. The renowned Gibbard-Satterthwaite theorem [11, 22] showed that if the preferences on the alternatives for agents are arbitrary, the only strategy-proof mechanism is the dictatorship when the number of alternatives are greater than two.

In real life, agent preferences on the locations are not arbitrary. In particular for the facility game over a real line, agents should have *single-peaked preferences*, where peaks are at agents' own locations. This kind of admissible preference was first discussed by Black [6]. Later, Moulin [19], Barberà and Jakson [4], and Sprumont [25] characterized the class of all strategy-proof mechanisms for the one-facility game in the real line. Interested readers may refer to the detailed survey by Barberà [3]. Notably, the characterization for the strategy-proof mechanisms with two or more facilities (even over a line) is wide open.

In additional to the social cost, Procaccia and Tennenholtz [21] and Alon et al. [1] also considered another optimization target, the maximum cost. They obtained lower and upper bounds for the approximation ratios of strategy-proof mechanisms for this target. Another extension of the facility games was studied in [21] and [17]. In this game, an agent may have more than one location and is aiming to minimize the overall cost of all the locations she have.

2. PRELIMINARIES

Let (Ω, d) be a metric space where $d : \Omega \times \Omega \to \mathbb{R}$ is the metric. The distance between any two points $x, y \in \Omega$ is $d(x, y)$. Recall that for all $x \in \Omega$, $d(x, x) = 0$.

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. The location reported by agent i is $x_i \in \Omega$. We denote $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ a *location profile*.

In the k-facility game, a *deterministic mechanism* outputs k facility locations according to a given location profile $\mathbf{x}$, and thus is a function $f : \Omega^n \to \Omega^k$. Assuming the set of facility locations to be $f(\mathbf{x}) = \{l_1, l_2, ...l_k\}$, the cost of agent i is her distance to the nearest facility:

$$\text{cost}(f(\mathbf{x}), x_i) = \min_{j=1,\cdots,k} \{d(l_j, x_i)\}.$$

A *randomized mechanism* is a function $f : \Omega^n \to \Delta(\Omega^k)$, where $\Delta(\Omega^k)$ is the set of distributions over Ω^k. The cost of agent i is now her expected cost over such distribution:

$$\text{cost}(f(\mathbf{x}), x_i) = \mathbb{E}_{\mathbf{l}\sim f(\mathbf{x})}\left[\min_{j=1,\cdots,k} \{d(l_j, x_i)\}\right].$$

Let $\mathbf{x}_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ be the location profile without agent i. We write $\mathbf{x} = \langle x_i, \mathbf{x}_{-i}\rangle$. Similarly, when $S \subset N$ is a set of agents, we denote $\mathbf{x}_{-S}$ the location profile of agents outside S. We write $\mathbf{x} = \langle \mathbf{x}_S, \mathbf{x}_{-S}\rangle$, the location profile satisfying that agents in S report locations $\mathbf{x}_S$ while other agents report locations $\mathbf{x}_{-S}$. For simplicity, we denote $f(x_i, \mathbf{x}_{-i}) = f(\langle x_i, \mathbf{x}_{-i}\rangle)$ and $f(\mathbf{x}_S, \mathbf{x}_{-S}) = f(\langle \mathbf{x}_S, \mathbf{x}_{-S}\rangle)$.

The *social cost* of a mechanism f on a location profile $\mathbf{x}$ is defined as the total cost of all n agents:

$$SC(f, \mathbf{x}) = \sum_{i=1}^{n} \text{cost}(f(\mathbf{x}), x_i)$$

We note that in the randomized case, this social cost is an expected value. For a location profile $\mathbf{x}$, denote $\text{OPT}(\mathbf{x})$ the optimal social cost. We say that a mechanism f has an approximation ratio γ, if for all profile $\mathbf{x} \in \Omega^n$,

$$SC(f, \mathbf{x}) \le \gamma \text{OPT}(\mathbf{x}).$$

In this paper, we stick to the case of $k = 2$ which we name it the *two-facility game*. Besides the general metric space, we also study two special cases: the *line* metric space and the *circle* metric space. The line metric is simply the Euclidean metric on the real line; the circle metric is defined as the length of the minor arc between any two points on $S^1 \subset \mathbb{R}^2$. Our definitions of line and circle are consistent with that in [1].

Now, we give formal definitions of *strategy-proofness* and *group strategy-proofness.*

Definition 2.1. *A mechanism is* strategy-proof *if no agent can benefit from misreporting her location. Formally, given agent i, profile $\mathbf{x} = \langle x_i, \mathbf{x}_{-i}\rangle \in \Omega^n$, and a misreported location $x'_i \in \Omega$, it holds that*

$$\text{cost}(f(x_i, \mathbf{x}_{-i}), x_i) \le \text{cost}(f(x'_i, \mathbf{x}_{-i}), x_i).$$

Definition 2.2. [2] *A mechanism is* group strategy-proof *if for any group of agents, at least one of them cannot benefit if they misreport simultaneously.*

Formally, given a non-empty set $S \subset N$, profile $\mathbf{x} = \langle \mathbf{x}_S, \mathbf{x}_{-S}\rangle \in \Omega^n$, and the misreported locations $\mathbf{x}'_S \in \Omega^{|S|}$, there exists $i \in S$, satisfying

$$\text{cost}(f(\mathbf{x}_S, \mathbf{x}_{-S}), x_i) \le \text{cost}(f(\mathbf{x}'_S, \mathbf{x}_{-S}), x_i).$$

2.1 Partial Group Strategy-Proofness

Inspired by the group strategy-proofness, we define the partial group strategy-proofness:

Definition 2.3. *A mechanism is* partial group strategy-proof *if for any group of agents at the same location, each individual cannot benefit if they misreport simultaneously.*

Formally, given a non-empty set $S \subset N$, profile $\mathbf{x} = \langle \mathbf{x}_S, \mathbf{x}_{-S}\rangle \in \Omega^n$ where $\mathbf{x}_S = (x, ..., x)$ for some $x \in \Omega$, and the misreported locations $\mathbf{x}'_S \in \Omega^{|S|}$, we have:

$$\text{cost}(f(\mathbf{x}_S, \mathbf{x}_{-S}), x) \le \text{cost}(f(\mathbf{x}'_S, \mathbf{x}_{-S}), x)$$

Intuitively, the definition says that a group of overlapping agents cannot "group-misreport" and benefit. By definition, we have the following:

group strategy-proofness
$\Rightarrow$ partial group strategy-proofness
$\Rightarrow$ strategy-proofness.

In the following, we show that one reversal direction also holds:

[2]Here we use the weak notion of group strategy-proofness which follows the definitions in [21, 1]. Some other work defines the strong group strategy-proofness by asking that it cannot be the case that all the deviating agents do not lose and at least one strictly gains.

Lemma 2.1. *In a k-facility game, a strategy-proof mechanism is also partial group strategy-proof.*

Proof. We embrace the same notations as in Definition 2.3. In addition, we let $S = \{s_1, s_2, \dots, s_l\}$, and x'_{s_i} be the misreported location for agent s_i in $\mathbf{x}'_S$. Consider the following sequence of profiles :

$$P_i(0 \le i \le l): \quad \begin{array}{l} s_j \text{ reports } x \text{ for } 1 \le j \le i; \\ s_j \text{ reports } x'_{s_i} \text{ for } i < j \le l; \\ \text{other agents report } \mathbf{x}_{-S}. \end{array}$$

By definition, we have

$$\text{cost}(f(P_0), x) = \text{cost}(f((x, ..., x), \mathbf{x}_{-S}), x)$$

and

$$\text{cost}(f(P_l), x) = \text{cost}(f(\mathbf{x}'_S, \mathbf{x}_{-S}), x).$$

We are to prove that $\text{cost}(f(P_0), x) \le \text{cost}(f(P_l), x)$.

In profile P_i where $1 \le i \le l$, agent s_i is at location x. We consider the scenario that agent s_i misreports to x'_{s_i}, and this is exactly P_{i-1}. By the strategy-proofness of f, agent s_i cannot benefit from this misreport: $\text{cost}(f(P_i), x)) \le \text{cost}(f(P_{i-1}), x)$. Summing up these inequalities for all $i = 1, 2, ..., l$, we complete the proof. □

We remark that our lower bound result in the next section will be proved with the aid of the notion of partial group strategy-proofness. The definition of partial group strategy-proof is not restricted to the two-facility game; or rather it also works for k-facility games for any k. This fact may be of independent interest.

3. LINEAR LOWER BOUND FOR DETERMINISTIC MECHANISMS

In this section, we give a linear lower bound of $\frac{n-1}{2}$ on the approximation ratio for deterministic strategy-proof mechanisms. This bound is constructed in the line metric space, which naturally extends to other more general metric spaces. The previous known lower bounds are only constants [21, 17].

For the two-facility game on the real line, choosing the leftmost and the rightmost points in the location profile is a deterministic strategy-proof mechanism with an approximation ratio of $n - 2$ [21]. Therefore, our lower bound implies that this simple mechanism is asymptotically optimal.

3.1 Image Set

We first explore some properties of the k-facility game. These properties will be used for our two-facility case, but may be of independent interest for further studies.

We define the concept of *image set*. For a given mechanism f, the image set of agent i with respect to a location profile $\mathbf{x}_{-i}$ is the set of all possible facility locations when agent i varies her reported location:

$$\text{I}_i(\mathbf{x}_{-i}) = \cup_{x_i \in \Omega} f(x_i, \mathbf{x}_{-i}).$$

The following lemma states that a strategy-proof mechanism f always outputs some location in $\text{I}_i(\mathbf{x}_{-i})$ that is closest to agent i. Intuitively, the image set represents agent i's power. If f outputs the best solution for agent i within her power, agent i does not have the incentive to lie.

Lemma 3.1. *Let f be a strategy-proof mechanism for the k-facility game, $\langle x_i, \mathbf{x}_{-i} \rangle \in \Omega^n$. We have:*

$$\text{cost}(f(x_i, \mathbf{x}_{-i}), x_i) = \inf_{y \in \text{I}_i(\mathbf{x}_{-i})} d(y, x_i).$$

Proof. We assume for contradiction that there exists $y^* \in \text{I}_i(\mathbf{x}_{-i})$ such that $d(y^*, x_i) < \text{cost}(f(x_i, \mathbf{x}_{-i}), x_i)$.

By the definition of image set, there exists x_i^* satisfying $y^* \in f(x_i^*, \mathbf{x}_{-i})$. Consider the scenario that agent i is at x_i. She can misreport to x_i^*, experiencing a lower cost of $d(y^*, x_i)$ than her current cost of $\text{cost}(f(x_i, \mathbf{x}_{-i}), x_i)$. This contradicts the assumption that f is strategy-proof. □

This lemma implies that if an agent misreports to one of the current facilities, this facility will stay at the same location. Formally, we have:

Corollary 3.2. *Let f be a strategy-proof mechanism for the k-facility game. Let $\mathbf{x} = \langle x_i, \mathbf{x}_{-i} \rangle$ be a location profile. If $z \in f(\mathbf{x})$, we must have $z \in f(z, \mathbf{x}_{-i})$.*

Proof. By the definition of image set, $z \in \text{I}_i(\mathbf{x}_{-i})$ because $z \in f(\mathbf{x}) = f(x_i, \mathbf{x}_{-i})$. According to Lemma 3.1, $\text{cost}(f(z, \mathbf{x}_{-i}), z) = \inf_{y \in \text{I}_i(\mathbf{x}_{-i})} d(y, z)$. But the right hand side is 0 since $z \in \text{I}_i(\mathbf{x}_{-i})$. This implies $z \in f(z, \mathbf{x}_{-i})$. □

The following result is another direct corollary of Lemma 3.1.

Corollary 3.3. *Let $\text{I}_i(\mathbf{x}_{-i})$ be an image set of a strategy-proof mechanism for the k-facility game in metric space (Ω, d). Then $\text{I}_i(\mathbf{x}_{-i})$ is a closed set of Ω under the topology induced by the metric $d(\cdot)$.*

Now we extend the definition of image set from single agent to the multi agent. Given mechanism f, we define the image set of S with respect to $\mathbf{x}_{-S}$ as follows:

$$\text{J}_S(\mathbf{x}_{-S}) = \bigcup_{\mathbf{x}_S \in \Omega^{|S|}} f(\mathbf{x}_S, \mathbf{x}_{-S}).$$

Using partial group strategy-proofness, Lemma 3.1, Corollary 3.2 and Corollary 3.3 have the corresponding multi-agent counterparts.

Lemma 3.4 (Extending 3.1). *Let f be a strategy-proof mechanism for the k-facility game. Let $S \subset N$ be a non-empty set of agents, $\mathbf{x}_S = (x, ..., x)$, and $\mathbf{x}_{-S} \in \Omega^{n-|S|}$. We have:*

$$\text{cost}(f(\mathbf{x}_S, \mathbf{x}_{-S}), x) = \inf_{y \in \text{J}_S(\mathbf{x}_{-S})} d(y, x).$$

Proof. Assume the statement is false, there exists $y^* \in \text{J}_S(\mathbf{x}_{-S})$ such that $d(y^*, x) < \text{cost}(f(\mathbf{x}_S, \mathbf{x}_{-S}), x)$.

By the definition of image set, there exists $\mathbf{x}'_S$ satisfying $y^* \in f(\mathbf{x}'_S, \mathbf{x}_{-S})$. By the partial group strategy-proofness (Lemma 2.1) of f, agents in S for profile $\langle \mathbf{x}_S, \mathbf{x}_{-S} \rangle$ cannot "group-misreport" to $\mathbf{x}'_S$ and benefit. Therefore, we have

$$\text{cost}(f(\mathbf{x}_S, \mathbf{x}_{-S}), x) \le \text{cost}(f(\mathbf{x}'_S, \mathbf{x}_{-S}), x) \le d(y^*, x_i),$$

resulting in a contradiction. □

Similarly, we have the following two corollaries.

Corollary 3.5 (Extending 3.2). *Let f be a strategy-proof mechanism for the k-facility game. Let $S \subset N$ be a set of agents, and $\mathbf{x}_{-S} \in \Omega^{n-|S|}$, we have:*

$$\forall x \in \text{J}_S(\mathbf{x}_{-S}),\ x \in f((x, ..., x), \mathbf{x}_{-S}).$$

Corollary 3.6 (Extending 3.3). *$\text{J}_S(\mathbf{x}_{-S})$ is closed in Ω.*

3.2 Proof of the Lower Bound

In this section we state and prove our main lower bound theorem.

THEOREM 3.7. *Any deterministic strategy-proof mechanism for the two-facility game in the line metric space has an approximation ratio of at least* $\frac{n-1}{2}$.

Our lower bound is obtained by a careful study of the behavior of any mechanism on the following set of profiles

$$\mathbf{x}(a,b) = (\underbrace{a, a, \ldots, a}_{(n-1)/2}, \underbrace{b, b, \ldots, b}_{(n-1)/2}, 1),$$

where $a \le b \le 1$ are two parameters. Intuitively, when $a = -1$ and $b = 0$, a mechanism with a good approximation ratio should allocate one facility near a and the other facility near b; when the distance between a and b is very small, it should allocate one facility near a (and hence b) and the other near 1. However, we will show that a strategy-proof mechanism cannot do well in both cases.

We notice that in $\mathbf{x}(a,b)$, $\frac{n-1}{2}$ agents are at a same location a and another $\frac{n-1}{2}$ agents are at a same location b. This configuration enables us to adopt the partial group strategy-proofness.

Let S_a (resp. S_b) be the $\frac{n-1}{2}$ agents at location a (resp. b). Then $\mathbf{x}_{-S_a}$ (resp. $\mathbf{x}_{-S_b}$) is the location profile that agents in S_b (resp. S_a) report b (resp. a) and the last agent reports 1. We define:

$$\mathrm{I}_a(b) = \mathrm{J}_{S_a}(\mathbf{x}_{-S_a}) = \mathrm{J}_{S_a}((b, ..., b, 1));$$

$$\mathrm{I}_b(a) = \mathrm{J}_{S_b}(\mathbf{x}_{-S_b}) = \mathrm{J}_{S_b}((a, ..., a, 1)).$$

LEMMA 3.8. *Let f be a deterministic strategy-proof mechanism for a line metric space with an approximation ratio smaller than $\frac{n-1}{2}$. Then $a \in f(\mathbf{x}(a,b))$ for all $a \le b \le 1$.*

PROOF. The lemma is obvious when $b = 1$. Consider $\mathrm{I}_a(b)$ for $b < 1$. We first show that $\mathrm{I}_a(b) \cap (-\infty, b) = (-\infty, b)$, by assuming for contradiction that there exists some $c < b$ satisfying $c \notin \mathrm{I}_a(b)$.

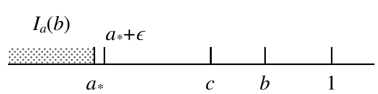

Figure 1: The definition of c, a_* and $a_* + \epsilon$

Notice that when $a \to -\infty$, any mechanism with a bounded approximation ratio will place a facility close to a and hence on the left side of c. This indicates that $\mathrm{I}_a(b) \cap (-\infty, c) \ne \emptyset$. Therefore $a_* = \sup_{x \in \mathrm{I}_a(b)}\{x < c\}$ is well defined. Since $\mathrm{I}_a(b)$ is closed according to Corollary 3.6, we have $a_* \in \mathrm{I}_a(b)$.

Now we have $a_* < c < b$, as shown in Figure 1. According to definitions above, we have $(a_*, c] \cap \mathrm{I}_a(b) = \emptyset$. For any $0 \le \epsilon < (c - a_*)/2$, the closest point to $a_* + \epsilon$ in the image set $\mathrm{I}_a(b)$ is a_* (this point is unique). Thus by Lemma 3.4, $a_* \in f(\mathbf{x}(a_* + \epsilon, b))$. We fix $\epsilon = \frac{c-a_*}{3} \le \frac{b-a_*}{3}$, and consider the following profile:

$$\mathbf{x}' = (\underbrace{a_* + \epsilon, a_* + \epsilon, \ldots, a_* + \epsilon}_{(n-1)/2}, \underbrace{b, b, \ldots, b}_{(n-1)/2}, a_*).$$

Using the fact that $a_* \in f(\mathbf{x}(a_* + \epsilon, b))$ and Corollary 3.2, we know $a_* \in f(\mathbf{x}')$. However, no matter where the second facility is placed by f, the social cost is at least $\frac{(n-1)\epsilon}{2}$. This contradicts that f has an approximation ratio smaller than $(n-1)/2$, because the optimal social cost in profile $\mathbf{x}'$ is only ϵ. In sum, we must have $\mathrm{I}_a(b) \cap (-\infty, b) = (-\infty, b)$.

Finally, using Corollary 3.5, it is clear that for any $a < b$, $a \in f(\mathbf{x}(a,b))$. For the case of $a = b$ the result is trivial. □

Using analogous techniques we can prove the following lemma, whose proof can be found in the appendix.

LEMMA 3.9. *Let f be a deterministic strategy-proof mechanism for the line metric space with an approximation ratio smaller than $\frac{n-1}{2}$. We have $b \in f(\mathbf{x}(a,b))$ for all $a \le b \le 1$.*

PROOF OF THEOREM 3.7. We consider profile

$$\tilde{\mathbf{x}} = (\underbrace{0, 0, \ldots, 0}_{(n-1)/2}, \underbrace{\frac{1}{n^2}, \frac{1}{n^2}, \ldots, \frac{1}{n^2}}_{(n-1)/2}, 1).$$

By Lemma 3.8 and 3.9, any strategy-proof mechanism f with an approximation ratio smaller than $\frac{n-1}{2}$ will place facilities at $\frac{1}{n^2}$ and 0, achieving a social cost of 1. However, the optimal social cost for $\tilde{\mathbf{x}}$ is only $\frac{1}{2n}$ by placing facilities at 0 and 1. This contradicts the assumption that f has an approximation ratio smaller than $\frac{n-1}{2}$. □

3.3 Discussions

Our lower bound is constructed in the line metric space, which directly applies to other general metric spaces. It also holds for any metric space which can be locally viewed as a line, such as the circle. On the line, there is an upper bound of $n - 2$, which asymptotically matches our lower bound. However, this lower bound may not be tight for more general metrics. For example, there is no known upper bound for deterministic mechanisms in metric spaces other than line and circle (to be shown in Section 5). It could be the case that the approximation ratio is actually unbounded for general metric spaces.

Our technique can be extended to show a linear lower bound for the k-facility game when $k > 2$. It is unknown whether this bound is tight even on the line. In particular, it remains an open question that whether a deterministic mechanism exists for three-facility games with any bounded approximation ratio even in the line metric space.

4. PROPORTIONAL MECHANISM

In the previous section, we proved that there is no deterministic strategy-proof mechanism with a good (sub-linear) approximation ratio. In this section, we propose the first randomized mechanism with a constant approximation ratio. Notice that the best known randomized mechanism [17] has an approximation ratio of $n/2$, and works only in the line metric space. Our mechanism works for general metric spaces.

Proportional Mechanism.
Given a profile $\mathbf{x} = (x_1, x_2, ..., x_n)$, the locations of the two facilities are decided by the following random process:

Round 1: *Choose agent i uniformly at random from N. The first facility l_1 is placed at x_i.*

Round 2: *Let $d_j = d(l_1, x_j)$ be the distance from agent j*

to the first facility l_1. Choose agent j with probability $\frac{d_j}{\sum_{k \in N} d_k}$. The second facility is then placed at x_j.[3]

The Proportional Mechanism always allocates facilities on the reported locations. The probability of the placement of the second facility is proportional to its distances to the first facility. This is where the name "Proportional" comes from.

The Proportional Mechanism has the following nice property. Every term in the expected cost has a form of $\frac{X}{Y}Z$, where X, Y, Z are some distances. $\frac{X}{Y}$ is a ratio which indicates a probability, and Z is a cost. However, we can also view $\frac{Z}{Y}$ as a ratio, and X as a cost. This small observation is used extensively both in the proof of strategy-proofness and the analysis of the approximation ratio.

4.1 Strategy-Proofness

THEOREM 4.1. *The Proportional Mechanism for the two-facility game is strategy-proof.*

PROOF. We use $\text{cost}_k(f(\mathbf{x}), x_i)$ to denote the expected cost of the agent i conditional on that the first facility is at x_k. It is clear that $\text{cost}_i(f(\mathbf{x}), x_i) = 0$. The total cost for agent i is

$$\text{cost}(f(\mathbf{x}), x_i) = \frac{1}{n}\sum_{k=1}^{n} \text{cost}_k(f(\mathbf{x}), x_i) = \frac{1}{n}\sum_{k \neq i} \text{cost}_k(f(\mathbf{x}), x_i).$$

Consider profile $\mathbf{x}' = \langle x_i', \mathbf{x}_{-i}\rangle$, in which agent i misreports her location from x_i to x_i'. To prove the strategy-proofness, it is sufficient to prove that for all $k \neq i$,

$$\text{cost}_k(f(\mathbf{x}'), x_i) \geq \text{cost}_k(f(\mathbf{x}), x_i).$$

Now we fix the first facility on x_k. We recall that $d_i = d(l_1, x_i) = d(x_k, x_i)$ and $\text{cost}_k(f(\mathbf{x}), x_i)$ is

$$\frac{\sum_{j=1}^{n} d_j \min\{d_i, d(x_i, x_j)\}}{\sum_{j=1}^{n} d_j} = \frac{\sum_{j \neq i} d_j \min\{d_i, d(x_i, x_j)\}}{\sum_{j=1}^{n} d_j}.$$

Let $d_i' = d(l_1, x_i')$. The cost of agent i if she misreports, i.e. $\text{cost}_k(f(\mathbf{x}'), x_i)$ is

$$\frac{\sum_{j \neq i} d_j \min\{d_i, d(x_i, x_j)\}}{\sum_{j=1}^{n} d_j + (d_i' - d_i)} + \frac{d_i' \min\{d_i, d(x_i, x_i')\}}{\sum_{j=1}^{n} d_j + (d_i' - d_i)}.$$

Comparing the above two expressions, we have the following relation:

$$\text{cost}_k(f(\mathbf{x}'), x_i) = \frac{\text{cost}_k(f(\mathbf{x}), x_i)\sum_{j=1}^{n} d_j}{\sum_{j=1}^{n} d_j + (d_i' - d_i)} + \frac{d_i' \min\{d_i, d(x_i, x_i')\}}{\sum_{j=1}^{n} d_j + (d_i' - d_i)}.$$

If $d_i' \leq d_i$, the first term on the right hand side is already greater than $\text{cost}_k(f(\mathbf{x}), x_i)$, while the second term is non-negative. Therefore we only need to consider the case that $d_i' > d_i$. We have,

$$\begin{aligned}&\text{cost}_k(f(\mathbf{x}'), x_i) - \text{cost}_k(f(\mathbf{x}), x_i)\\ =&\frac{-(d_i' - d_i)\text{cost}_k(f(\mathbf{x}), x_i)}{\sum_{j=1}^{n} d_j + (d_i' - d_i)} + \frac{d_i' \min\{d_i, d(x_i, x_i')\}}{\sum_{j=1}^{n} d_j + (d_i' - d_i)}.\end{aligned}$$

So it is sufficient to show that

$$d_i' \min\{d_i, d(x_i, x_i')\} - (d_i' - d_i)\text{cost}_k(f(\mathbf{x}), x_i) \geq 0. \quad (1)$$

We prove this for two cases.

- If $\min\{d_i, d(x_i, x_i')\} = d_i$, inequality (1) holds because $d_i' \geq d_i' - d_i$ and $d_i \geq \text{cost}_k(f(\mathbf{x}), x_i)$. Here the latter holds because agent i can at least choose the first facility, which is at x_k, to serve him with cost $d(x_i, x_k) = d_i$.
- If $\min\{d_i, d(x_i, x_i')\} = d(x_i, x_i')$, inequality (1) holds because $d_i' \geq d_i \geq \text{cost}_k(f(\mathbf{x}), x_i)$ and $d(x_i, x_i') \geq d_i' - d_i$. Here the latter is due to the triangle inequality in the metric space (Ω, d) since $d_i' = d(l_1, x_i')$ and $d_i = d(l_1, x_i)$.

This completes the proof. □

From the above proof, we can see that our Proportional Mechanism is strategy-proof even in a slightly stronger sense. An agent does not have the incentive to lie even if she has seen the random bits in the first round of the mechanism.

4.2 Approximation Ratio for Social Cost

In this section, we estimate the approximation ratio of our Proportional Mechanism in general metric spaces and prove the following theorem.

THEOREM 4.2. *The approximation ratio of the Proportional Mechanism for the two-facility game is at most 4 for any metric space.*

For a location profile $\mathbf{x}$, let f_α and f_β be the locations of the two facilities in one optimal solution. Let α be the set of agents that are strictly closer to f_α than to f_β, and β be the rest. We use OPT_α to denote the summation of costs of agents in α and OPT_β the summation of costs of agents in β. Clearly, $\text{OPT} = \text{OPT}_\alpha + \text{OPT}_\beta$.

Similarly, let cost_α (resp. cost_β) be the total costs of agents in α (resp. β), assuming facilities are chosen according to our Proportional Mechanism. Let F_α (resp. F_β) be the event that the agent chosen by the mechanism at the first round is in α (resp. β). Since our mechanism is randomized, both cost_α and cost_β are random variables. We need to bound the expected cost of our mechanism, which is $\mathbb{E}[\text{cost}_\alpha + \text{cost}_\beta]$. F_α and F_β are two exclusive random events, which form a partition of the whole probabilistic space. Therefore, the cost $\mathbb{E}[\text{cost}_\alpha + \text{cost}_\beta]$ is equal to

$$\Pr(F_\alpha)\mathbb{E}[(\text{cost}_\alpha + \text{cost}_\beta)|F_\alpha] + \Pr(F_\beta)\mathbb{E}[(\text{cost}_\alpha + \text{cost}_\beta)|F_\beta].$$

Next two lemmas bound the expected values $\mathbb{E}[(\text{cost}_\alpha + \text{cost}_\beta)|F_\alpha] = \mathbb{E}[\text{cost}_\alpha|F_\alpha] + \mathbb{E}[\text{cost}_\beta|F_\alpha]$. Similar results can be deduced for $\mathbb{E}[(\text{cost}_\alpha + \text{cost}_\beta)|F_\beta]$.

LEMMA 4.3. $\mathbb{E}[\text{cost}_\alpha|F_\alpha] \leq 2\text{OPT}_\alpha$

PROOF. Note that $\mathbb{E}[\text{cost}_\alpha|F_\alpha] \leq \frac{1}{|\alpha|}\sum_{i\in\alpha}\sum_{j\in\alpha} d(x_i, x_j)$ if we completely ignore the second facility. Since $\text{OPT}_\alpha = \sum_{i\in\alpha} d(x_i, f_\alpha)$, by triangle inequality, we have $|\alpha| \cdot \text{OPT}_\alpha = \sum_{i\in\alpha} |\alpha| \cdot d(x_i, f_\alpha) = \frac{1}{2}\sum_{i\in\alpha}\sum_{j\in\alpha}(d(x_i, f_\alpha) + d(x_j, f_\alpha)) \geq \frac{1}{2}\sum_{i\in\alpha}\sum_{j\in\alpha} d(x_i, x_j)$. The lemma follows. □

LEMMA 4.4. $\mathbb{E}[\text{cost}_\beta|F_\alpha] \leq 2\text{OPT}_\alpha + 4\text{OPT}_\beta$.

PROOF. We define

$$\text{cost}_\beta^{k,i} = \sum_{j\in\beta} \min\{d(x_k, x_j), d(x_i, x_j)\}$$

to be the cost of the agents in β given the condition that the first chosen agent is k and the second one is i in the

[3]If all the agents report the same location, our mechanism places the second facility also on this location.

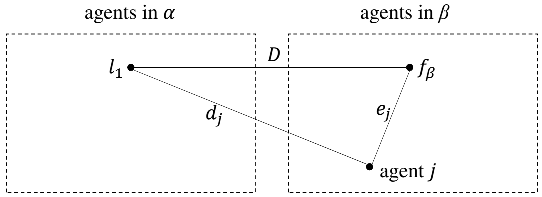

Figure 2: Definitions of D, d_i and e_i

Proportional Mechanism. We denote $P(i|k)$ as the probability that the second chosen agent is i conditional on that the first chosen one is k. Then we have,

$$\mathbb{E}[\text{cost}_\beta|F_\alpha] = \sum_{k\in\alpha}\frac{1}{|\alpha|}\sum_{i\in\alpha}\text{cost}_\beta^{k,i}\cdot P(i|k) + \sum_{k\in\alpha}\frac{1}{|\alpha|}\sum_{i\in\beta}\text{cost}_\beta^{k,i}\cdot P(i|k). \tag{2}$$

For first term in Eq.(2), we ignore the second facility and bound the total costs of agents in β using their distances to $l_1(= x_k)$.

$$\begin{aligned}&\sum_{k\in\alpha}\frac{1}{|\alpha|}\sum_{i\in\alpha}\text{cost}_\beta^{k,i}P(i|k)\\ &\le \sum_{i\in\alpha}\frac{1}{|\alpha|}\sum_{k\in\alpha}\left(\sum_{j\in\beta}d(x_k,x_j)\right)\frac{d(x_k,x_i)}{\sum_{j\in N}d(x_k,x_j)}\\ &\le \sum_{i\in\alpha}\frac{1}{|\alpha|}\sum_{k\in\alpha}d(x_k,x_i)\le 2\text{OPT}_\alpha\end{aligned} \tag{3}$$

where the last inequality is due to Lemma 4.3.

For the second term in Eq.(2), we will bound the internal summation for any fixed $k \in \alpha$. So we fixed the first facility $l_1(= x_k)$ and denote $d_j = d(l_1, x_j)$. As shown in Figure 2, we define $D = d(l_1, f_\beta)$ to be the distance between l_1 and the optimal facility in β. Furthermore, for agent j in β, let $e_j = d(f_\beta, x_j)$ be the distance from agent j to the optimal facility in β, and denote $s_j = d_j - e_j$. It is clear that $\text{OPT}_\beta = \sum_{j\in\beta} e_j$.

Notice that s_j can be negative by our definition. However, we always have $\sum_{j\in\beta} s_j \ge 0$, since otherwise l_1 is a strictly better facility location for agents in β than f_β, contradicting the optimality of f_β.

Now we calculate the total costs for agents in β:

$$\begin{aligned}&\sum_{i\in\beta}\text{cost}_\beta^{k,i}P(i|k)\\ &= \sum_{i\in\beta}\left(\sum_{j\in\beta}\min\{d_j, d(x_i,x_j)\}\right)\frac{d_i}{\sum_{j\in N}d_j}\\ &= \sum_{i\in\beta}\frac{e_i+s_i}{\sum_{j\in\beta}e_j+s_j}\left(\sum_{j\in\beta}\min\{e_j+s_j, d(x_i,x_j)\}\right).\end{aligned} \tag{4}$$

By triangle inequality, we have $d(x_i, x_j) \le e_j + e_i$, and we continue to bound the above equation:

$$\begin{aligned}&\sum_{i\in\beta}\text{cost}_\beta^{k,i}P(i|k)\\ \le& \sum_{i\in\beta}\frac{e_i+s_i}{\sum_{j\in\beta}e_j+s_j}\sum_{j\in\beta}\min\{e_j+s_j, e_j+e_i\})\\ =& \sum_{i\in\beta}\frac{e_i+s_i}{\sum_{j\in\beta}e_j+s_j}\sum_{j\in\beta}e_j\\ &+\sum_{i\in\beta}\frac{e_i}{\sum_{j\in\beta}e_j+s_j}\sum_{j\in\beta}\min\{s_j,e_i\}\\ &+\sum_{i\in\beta}\frac{s_i}{\sum_{j\in\beta}e_j+s_j}\sum_{j\in\beta}\min\{s_j,e_i\}.\end{aligned} \tag{5}$$

The first term of the last summation is exactly $\sum_{j\in\beta} e_j = \text{OPT}_\beta$. For the second term, we relax $\min\{s_j, e_i\}$ to s_j. Because $\sum_{j\in\beta} e_j + s_j \ge \sum_{j\in\beta} s_j$, the second term is bounded by $\sum_{j\in\beta} e_j = \text{OPT}_\beta$.

For the third term, we relax $\min\{s_j, e_i\}$ to e_i. By triangle inequality, we have $e_j + D \ge d_j \Rightarrow s_j \le D$ and $d_j + e_j \ge D$. Therefore,

$$\begin{aligned}&\sum_{i\in\beta}\frac{s_i}{\sum_{j\in\beta}e_j+s_j}\sum_{j\in\beta}\min\{s_j,e_i\}\\ \le& \sum_{i\in\beta}\frac{s_i|\beta|e_i}{\sum_{j\in\beta}e_j+s_j}\\ \le& \sum_{i\in\beta}e_i\frac{|\beta|\cdot D}{\sum_{j\in\beta}e_j+s_j}\le 2\text{OPT}_\beta,\end{aligned} \tag{6}$$

where the last inequality is because (using the fact that $\sum_{j\in\beta} s_j \ge 0$)

$$\sum_{j\in\beta}2e_j+2s_j \ge \sum_{j\in\beta}2e_j+s_j = \sum_{j\in\beta}d_j+e_j \ge |\beta|\cdot D.$$

To put things together, we have

$$\sum_{i\in\beta}\text{cost}_\beta^{k,i}P(i|k)\le 4\text{OPT}_\beta. \tag{7}$$

Substituting Eq. (3) and Eq. (7) to Eq. (2), we have $\mathbb{E}[\text{cost}_\beta|F_\alpha] \le 2\text{OPT}_\alpha + 4\text{OPT}_\beta$. This completes the proof. $\square$

We are ready to prove the main theorem of this section.

Proof of Theorem 4.2. The theorem follows by the following chain of inequalities.

$$\begin{aligned}&\mathbb{E}[\text{cost}_\alpha+\text{cost}_\beta]\\ &\le \max\{\mathbb{E}[(\text{cost}_\alpha+\text{cost}_\beta)|F_\alpha], \mathbb{E}[(\text{cost}_\alpha+\text{cost}_\beta)|F_\beta]\}\\ &= \max\{\mathbb{E}[\text{cost}_\alpha|F_\alpha]+\mathbb{E}[\text{cost}_\beta|F_\alpha], \mathbb{E}[\text{cost}_\alpha|F_\beta]+\mathbb{E}[\text{cost}_\beta|F_\beta]\}\\ &\le \max\{2\text{OPT}_\alpha+2\text{OPT}_\alpha+4\text{OPT}_\beta,\\ &\quad 2\text{OPT}_\beta+4\text{OPT}_\alpha+2\text{OPT}_\beta\}\\ &= 4(\text{OPT}_\alpha+\text{OPT}_\beta) = 4\text{OPT}.\end{aligned}$$

in which the bounds of $\mathbb{E}[\text{cost}_\alpha|F_\beta]$ and $\mathbb{E}[\text{cost}_\beta|F_\beta]$ are due to the symmetric versions of Lemma 4.3 and Lemma 4.4. $\square$

4.3 Discussion

It is worth noting that this upper bound of 4 is tight for our Proportional Mechanism even for the line metric space. Consider the location profile $\mathbf{x} = (\epsilon, 0, 0, ...0, 1)$, it can be shown that its approximation ratio tends to 4 as the number of agents is sufficiently large and $\epsilon \to 0$.

We note that the Proportional Mechanism is not group strategy-proof. It would be interesting if one can find a group strategy-proof mechanism with a constant approximation ratio.

We also examine two possible extensions of our Proportional Mechanism to the three-facility game. The first is to allocate the first two facilities the same as this section, but the third one in some agent w.p. proportional to her minimal distance to the first two facilities. Unfortunately we have found a non-trivial counter-example and shown that this mechanism is not strategy-proof. [4]

Another extension is a strategy-proof three-facility mechanism on the real line. The first two facilities are located at the leftmost and the rightmost reported locations. For the third facility, it is randomly chosen among the rest of the agents w.p. proportional to their minimal distances to the first two facilities. This mechanism guarantees a linear approximation ratio.

5. MECHANISM FOR CIRCLE

In this section, we consider the circle metric space (S^1, d), where $S^1 \subset \mathbb{R}^2$ is a circle in the two dimensional Euclidean space and the distance $d(x, y)$ for $x, y \in S^1$ is the length of the minor arc spanned by x and y. We can normalize the circle so that its circumference is 1. Notice that the $\frac{n-1}{2}$ deterministic lower bound in Section 3 can still be applied here, because a circle can be locally viewed as a line. Now we give a deterministic group strategy-proof mechanism with an approximation ratio of $n-1$. This is tight up to a constant factor of 2.

Circle Mechanism.
Given profile $\mathbf{x} = (x_1, x_2, ...x_n)$, the first facility is allocated at x_1, the location of the first agent. As shown in Figure 3 (a), we denote $\hat{x}_1$ the antipodal of x_1, and there form two semi-circles with x_1 and $\hat{x}_1$ as endpoints. We call one of the semi-circle the left circle $\mathfrak{L}$ *and the other the* right circle $\mathfrak{R}$ [5]*. Let A and B be the set of agents on $\mathfrak{L}$ and $\mathfrak{R}$ respectively. We assume agents at location x_1 and $\hat{x}_1$ (if any) appear in only A, and thus $A \cap B = \emptyset$. Define $d_A = \max_{i \in A} d(x_1, x_i)$ and $d_B = \max_{i \in B} d(x_1, x_i)$ (if B is empty, let $d_B = 0$). We allocate the second facility as follows:*

- *If $d_A < d_B$, facility l_2 is placed on $\mathfrak{R}$ with distance $\min\{\max\{d_B, 2d_A\}, 1/2\}$ to l_1.*
- *If $d_A \geq d_B$, facility l_2 is placed on $\mathfrak{L}$ with distance $\min\{\max\{d_A, 2d_B\}, 1/2\}$ to l_1.*

In this mechanism, the first facility is always allocated at the location of the first agent as a dictator. Let us break the circle at point $\hat{x}_1$ to make it a straight line and think the location of first agent as the origin. In this way, we can understand the intuition of the mechanism more clearly.

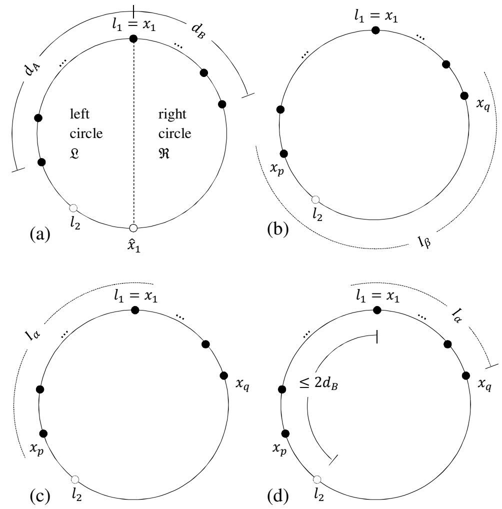

Figure 3: Mechanism on the Circle

After breaking the circle into a line, the coordinate of the rightmost (resp. leftmost) agent is d_B(resp. $-d_A$). If the distance from the rightmost agent to the origin is larger ($d_B > d_A$), we put the second facility on the right side at location $\max\{d_B, 2d_A\}$. Otherwise, we put the second facility on the left side at location $-\max\{d_A, 2d_B\}$. We can verify that the this line mechanism is group strategy-proof and has a linear approximation ratio.

However, when we transfer it back to the circle case, the location $\max\{d_B, 2d_A\}$ (resp. $-\max\{d_A, 2d_B\}$) may go across $\hat{x}_1$ to the left circle $\mathfrak{L}$ (resp. right circle $\mathfrak{R}$), which breaks the strategy-proofness. Therefore we put a cutoff at $\hat{x}_1$ for the circle mechanism, which means that we allocate the second facility at exactly $\hat{x}_1$ if $\max\{d_B, 2d_A\}$ is greater than $\frac{1}{2}$ (resp. if $-\max\{d_A, 2d_B\}$ is smaller than $-1/2$).

In the following proof, we shall keep this line interpretation in mind. For example, we call the agent farthest from l_1 in A the *leftmost agent* and the agent farthest from l_1 in B the *rightmost agent*.

5.1 Group Strategy-Proofness

THEOREM 5.1. *The Circle Mechanism is group strategy-proof.*

PROOF. We assume for contradiction that the Circle Mechanism f is not group strategy-proof. Then there exists a profile $\mathbf{x} = (x_1, x_2, ...x_n)$, a group of agents $S \subset N$ and their misreported locations $\mathbf{x}'_S$ such that for every agent $i \in S$, it is better off by the collusion, i.e.

$$\text{cost}(f(\mathbf{x}'_S, \mathbf{x}_{-S}), x_i) < \text{cost}(f(\mathbf{x}), x_i).$$

Without loss of generality, we assume $d_A \geq d_B$ for the given profile $\mathbf{x}$, and the case $d_A < d_B$ is similar. So l_2 lies on $\mathfrak{L}$ in $f(\mathbf{x})$.

The cost for the first agent is 0 in $f(\mathbf{x})$, so she cannot reduce her cost by any means and hence $1 \notin S$. This tells us that l_1 is still located as x_1 in $f(\mathbf{x}'_S, \mathbf{x}_{-S})$, and we assume $f(\mathbf{x}'_S, \mathbf{x}_{-S}) = \{l_1, l'_2\}$. We denote by C_1 the arc from l_1 to l_2 in an anti clockwise direction and by C_2 the arc from l_1 to l_2 in an clockwise direction. Then all agents in A are on C_1 and all agents in B are on C_2.

[4]This counter-example is as follows: there exist n_0 agents at location 0, n_1 agents at location 1, n_2 agents at location $1+x$ and 1 agent at location $1+x+y$. Here n_0 is sufficiently large such that we can assume the first facility l_1 to be always located at 0. In this configuration, let $y = 100$, $x = 10^5$, $n_1 = 50$ and $n_2 = 4$. After a careful calculation one may find out that the agent at location 1 may have the incentive to misreport to location $1 + x$.

[5]x_1 and $\hat{x}_1$ are assumed to be in both $\mathfrak{L}$ and $\mathfrak{R}$.

Obviously, l_2' can not be at l_1 or l_2 because otherwise no agent is better off. Therefore we have the following two cases:

Case 1: $l_2' \in C_1$**.** We first see that no agent in B can benefit from this misreport, because for an agent in B, either l_1 or l_2 will be her closer facility than the new l_2'. Therefore we have $S \subset A$ and $d_B' \geq d_B$.

Now, the colluded agents are all on C_1, and to benefit themselves, l_2' must still lie on C_1 with $d(l_1, l_2')$ strictly smaller than $d(l_1, l_2)$. This happens only when $d_A' < d_A$ according to our mechanism because agents in B do not lie. To have this, the leftmost agent in A must be in S and lie. Call this agent x_p. We cannot have $l_2 = x_p$ because otherwise agent p has already experienced a zero cost and has no incentive to lie. So we have $l_2 \neq x_p$. In this case, we have

$$d(l_1, l_2') \geq \min\{2d_B', 1/2\} \geq \min\{2d_B, 1/2\} = d(l_1, l_2),$$

contradicting our assumption that $d(l_1, l_2') < d(l_1, l_2)$.

Case 2: $l_2' \in C_2$**.** For similar reason as Case 1, no agent in A can benefit from the misreport, and thus $S \subset B$. As a result, $d_A' \geq d_A$. We further discuss three subcases regarding the location of l_2 and l_2'.

Subcase 2.1: $l_2 = \hat{x}_1$**.** To result in $l_2 = \hat{x}_1$, either $d_A = \frac{1}{2}$ or $d_B \geq \frac{1}{4}$.

If $d_A = \frac{1}{2}$, we must have $l_2' = l_2$ because agents in A do not lie. No agent can benefit in this scenario. If $d_B \geq \frac{1}{4}$, we have $d_A' \geq d_A \geq d_B \geq \frac{1}{4}$. To benefit themselves, l_2' must lie on the right circle $\mathfrak{R}$ because all the colluded agents are in B. But this cannot be the case since

$$\min\{\max\{d_B', 2d_A'\}, 1/2\} \geq \min\{2d_A', 1/2\} = \frac{1}{2}.$$

Subcase 2.2: $l_2 \neq \hat{x}_1$ **and** l_2' **is on** $\mathfrak{L}$ **(including** $\hat{x}_1$**).** Since $l_2 \neq \hat{x}_1$, we have $d_B < \frac{1}{4}$. So the distance from any agent $j \in B$ to l_2' is at least $d(x_j, l_1)$ because $l_2' \in \mathfrak{L}$. It is clear that any agent in B cannot thus benefit because her closest facility is still l_1.

Subcase 2.3: $l_2 \neq \hat{x}_1$ **and** l_2' **is on** $\mathfrak{R}$ **(excluding** $\hat{x}_1$**).** Then we have $d_B < \frac{1}{4}$ and $d_A \leq d_A' < \frac{1}{4}$. So for any agent $k \in B$, $d(x_k, l_2')$ is at least $2d_A' - d_B$, which is at least d_B since $d_B \leq d_A \leq d_A'$. This is already larger than or equal to its distance to the first facility which is d_B. This is a contradiction.

The theorem follows. □

5.2 Approximation Ratio for Social Cost

THEOREM 5.2. *The approximation ratio of the Circle Mechanism is at most* $n - 1$.

PROOF. For a given profile $\mathbf{x}$, consider the optimal solution using notations α and β defined in Section 4.2. We denote I_α the minimal arc covering all agents in α, and I_β the minimal arc covering all agents in β. It can be easily verified that $I_\alpha \cap I_\beta = \emptyset$. Let $|I_\alpha|$ be the length of I_α and $|I_\beta|$ be the length of I_β. Obviously OPT $\geq |I_\alpha| + |I_\beta|$.

Without loss of generality, we assume $l_1 = x_1 \in I_\alpha$ and $d_A \geq d_B$, so the second facility $l_2 \in \mathfrak{L}$ according to our mechanism.

Similar to Section 4.2, we let $\text{cost}_\alpha = \sum_{i \in \alpha} \text{cost}(f(\mathbf{x}), x_i)$ be the summation of costs of agents in α, and $\text{cost}_\beta = \sum_{i \in \beta} \text{cost}(f(\mathbf{x}), x_i)$. It is clear that $\text{cost}_\alpha \leq (|\alpha| - 1)\text{OPT}$, because $l_1 \in I_\alpha$ and any agent in α is at most $|I_\alpha| \leq \text{OPT}$ far from l_1, except $x_1 = l_1$ itself who has zero cost. Next we are to prove that $\text{cost}_\beta \leq |\beta|\text{OPT}$, which is enough to show our $n-1$ upper bound because $\text{cost} = \text{cost}_\alpha + \text{cost}_\beta \leq (n-1)\text{OPT}$.

If $l_2 \in I_\beta$, the distance from each agent in β to its closest facility is at most $|I_\beta|$. Thus, $\text{cost}_\beta \leq |\beta||I_\beta| \leq |\beta|\text{OPT}$.

If $l_2 \notin I_\beta$, let p be the leftmost agent on $\mathfrak{L}$, and q be the rightmost agent on $\mathfrak{R}$. We know that $d(l_1, x_p) = d_A$ and $d(l_1, x_q) = d_B$. If both x_p, x_q are in I_β (Figure 3 (b)), we will have a contradiction: according to the mechanism, l_1 and l_2 are on different arcs with x_p and x_q as endpoints, but I_β which contains both x_p and x_q must contain either l_1 or l_2. This contradicts the assumption of $l_1 \in I_\alpha$ or $l_2 \notin I_\beta$ respectively. Therefore, at least one of $x_p \in I_\alpha$ and $x_q \in I_\alpha$ hold.

- If $x_p \in I_\alpha$ (Figure 3 (c)), $|I_\alpha| \geq d_A$ because both x_1 and x_p are in. But each agent in β is at most $d_B \leq d_A$ far from l_1. This implies
$$\text{cost}_\beta \leq |\beta|d_A \leq |\beta||I_\alpha| \leq |\beta|\text{OPT}$$
- If $x_p \notin I_\alpha$, we have $x_q \in I_\alpha$ (Figure 3 (d)). We have $|I_\alpha| \geq d_B$ because both x_1 and x_q are in. So all agents in β are located on $\mathfrak{L}$, and thus each agent in β is no more than $d(l_1, l_2)/2$ far from either l_1 or l_2. Furthermore, based on the facts of $l_2 \notin I_\beta$ and $x_p \in I_\beta$, we deduce that $l_2 \neq x_p$. In this case, $d(l_1, l_2) \leq 2d_B$ according to the mechanism. In sum, we still have
$$\text{cost}_\beta \leq |\beta|\frac{d(l_1, l_2)}{2} \leq |\beta|d_B \leq |\beta||I_\alpha| \leq |\beta|\text{OPT}.$$

This completes the proof. □

5.3 Discussion

We remark here that the approximation ratio of the Circle Mechanism cannot be improved. Consider the profile $\mathbf{x} = (x_1, x_2, ..., x_n)$, where $d(x_1, x_2) = d(x_1, x_3) = 0.1$ and $x_3 = x_4 = ... = x_n$. But x_2 and x_3 are on different sides of x_1. In this case, OPT $= 0.1$ but the mechanism will give cost $= 0.1(n-1)$.

As noted, the mechanism here is motivated by the mechanism on a line. We do not know a mechanism with a bounded ratio for any slightly more complicated metric space. For example, for a star with three branches, we do not know how to extend our Circle Mechanism to this case. Furthermore, we can prove that if we fix the first facility as a dictator, no mechanism has a bounded ratio.

6. OPEN PROBLEMS AND DISCUSSION

In this section, we summarize some open problems related to this work.

1. The first remaining problem is to close the constant gaps both for deterministic (between $n-2$ and $\frac{n-1}{2}$)

and randomized (between 4 and 1.045) mechanisms in the line metric space.

2. It is interesting to explore deterministic mechanisms for the general metric spaces or the special metric spaces other than line or circle. To design a deterministic mechanism with any bounded ratio would be instructive. It is also possible that one can show that the approximation ratio is actually unbounded.

3. As noted in the paper, our Proportional Mechanism is not group strategy-proof. It remains open to provide a group strategy-proof randomized mechanism with a constant approximation ratio.

4. Another natural extension is to consider the game with three or more facilities. Our linear lower bound for deterministic mechanisms can be easily extended to more facilities. However, no deterministic mechanism with any bounded ratio has been known yet even for the line. It is significant if one can provide such a mechanism or prove that it does not exist. For a randomized setting, we can give a mechanism with a linear approximation ratio for the three-facility game in the line metric space. It would be very interesting to explore whether we can still get mechanisms with constant approximation ratio for games with more facilities.

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Appendix: Proof of Lemma 3.9

The lemma is obvious when $a = 1$. Consider $I_b(a)$ for $a < 1$. We first show that $I_b(a) \cap (a, 1) \neq \emptyset$.

If $I_b(a) \cap (a, 1) = \emptyset$, we consider the profile $\mathbf{x}(a, \frac{a+1}{2})$. The cost for agents at location $\frac{a+1}{2}$ is at least $\frac{1-a}{2}$ because $I_b(a) \cap (a, 1) = \emptyset$. Therefore, the social cost is at least $\frac{n-1}{2} \cdot \frac{1-a}{2}$, while the optimal social cost is only $\frac{1-a}{2}$. This contradicts the assumption that the approximation ratio of f is smaller than $\frac{n-1}{2}$. For such reason, there exists some $c_0 \in I_b(a) \cap (a, 1)$.

We then show that $I_b(a) \cap (a, 1) = (a, 1)$. Assume for contradiction that there exists $c \in (a, 1)$ satisfying $c \notin I_b(a)$. Obviously, $c \neq c_0$.

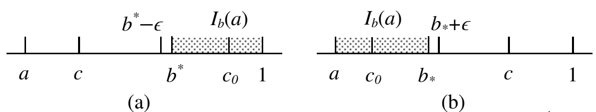

Figure 4: The definition of c_0, c, b_* and b^*

If $c < c_0$ (Figure 4 (a)), we define $b^* = \inf_{x \in I_b(a)}\{x > c\}$. Since $I_b(a)$ is closed (Corollary 3.6), we have $b^* \in I_b(a)$ and $a < c < b^* \leq c_0 < 1$.

According to definitions above, we have $[c, b^*) \cap \mathrm{I}_b(a) = \emptyset$. For any $0 \leq \epsilon < (b^* - c)/2$, the closest point to $b^* - \epsilon$ in the image set $\mathrm{I}_b(a)$ is b^* (this point is unique). Thus by Lemma 3.4, $b^* \in f(\mathbf{x}(b^* - \epsilon, b))$. Now fix $\epsilon = \frac{b^*-c}{3} < \frac{b^*-a}{3}$, and consider the following profile:

$$\mathbf{x}' = (\underbrace{a, a, \ldots, a}_{(n-1)/2}, \underbrace{b^* - \epsilon, b^* - \epsilon, \ldots, b^* - \epsilon}_{(n-1)/2}, b^*).$$

Using the fact that $b^* \in f(\mathbf{x}(b^* - \epsilon, b))$ and Corollary 3.2, we know $b^* \in f(\mathbf{x}')$. However, no matter where the second facility is located by f, the social cost is at least $\frac{(n-1)\epsilon}{2}$, contradicting that f has an approximation ratio smaller than $(n-1)/2$, since the optimal social cost in profile $\mathbf{x}'$ is only ϵ.

If $c > c_0$ (Figure 4 (b)), we define $b_* = \inf_{x \in I_b(a)}\{x < c\}$. Since $I_b(a)$ is closed, we have $b_* \in I_b(a)$ and $a < c_0 \leq b_* <= c < 1$.

According to definitions above, we have $(b_*, c] \cap \mathrm{I}_b(a) = \emptyset$. For any $0 \leq \epsilon < (c - b_*)/2$, the closest point to $b_* + \epsilon$ in the image set $\mathrm{I}_b(a)$ is b_* (this point is unique). Thus by Lemma 3.4, $b_* \in f(\mathbf{x}(b_* + \epsilon, b))$. Now fix $\epsilon = \min\{\frac{c-b_*}{3}, \frac{b_*-a}{3}\}$, and consider the following profile:

$$\mathbf{x}'' = (\underbrace{a, a, \ldots, a}_{(n-1)/2}, \underbrace{b_* + \epsilon, b_* + \epsilon, \ldots, b_* + \epsilon}_{(n-1)/2}, b_*).$$

Using the fact that $b_* \in f(\mathbf{x}(b_* + \epsilon, b))$ and Corollary 3.2, we know $b_* \in f(\mathbf{x}'')$. However, no matter where the second facility is located by f, the social cost is at least $\frac{(n-1)\epsilon}{2}$, contradicting that f has an approximation ratio smaller than $(n-1)/2$, since the optimal social cost in profile $\mathbf{x}''$ is only ϵ.

To sum up, we just proved that $I_b(a) \cap (a, 1) = (a, 1)$. According to Corollary 3.5, we have that $b \in f(\mathbf{x}(a, b))$ for all $a \leq b \leq 1$. For the case that $b = a$ or $b = 1$ the result is trivial.